INTENSITY CALCULATIONS ALONG A SINGLE RAY

by

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ABSTRACT

A method is presented for the calculation of sound intensity using only one ray. The sound velocity profile is assumed to be continuous up to the second derivative and it is supposed that the ray equations are solved numerically. An example is given.

INTRODUCTION

In this paper a method is presented for the calculation of the spreading loss along a ray, using only that particular ray. (The medium is assumed to be lossless, and so the intensity is equal to the spreading loss.) It appears that the spreading loss may be expressed by two different formulae, one not usable near any point where the first derivative of the sound velocity becomes zero, the other not near a turning point of the ray. By switching from one formula to the other near these critical points, these difficulties are avoided. It will be assumed that the sound-velocity profile is a continuous, differentiable curve, but not expressible in one mathematical formula, so that the ray-differential equations have to be solved numerically.

After a short introduction, in which some basic formulae used in the paper are given, the two intensity formulae will be discussed. Finally the method is illustrated with a computer result.

For the ray path shown in Fig. 1, the basic equations, when there is no z-dependence of the sound velocity c, are [Ref. 1]

$$\frac{d}{ds}\frac{1}{c}\frac{dx}{ds} = \frac{\partial}{\partial x}\frac{1}{c} \qquad \text{[Eq. 1a]}$$

$$\frac{d}{ds}\frac{1}{c}\frac{dy}{ds} = \frac{\partial}{\partial y}\frac{1}{c} \qquad \text{[Eq. 1b]}$$

For a velocity profile dependent only on y, this reduces to

$$\frac{dx}{ds} = c \cdot A \qquad \text{(Snell's law)} \qquad \text{[Eq. 2a]}$$

$$\frac{d}{ds}\frac{1}{c}\frac{dy}{ds} = -\frac{c'}{c^2} \qquad \text{[Eq. 2b]}$$

where A is the ray constant, the prime denotes the derivative with respect to y, and ds is the curvilinear parameter along the ray. From Fig. 1, we have

$$\frac{dy}{dx} = \tan\theta \, . \qquad \text{[Eq. 3]}$$

From Eq. 3 and eliminating ds from Eq. 2 it is easy to derive

$$\frac{d^2y}{dx^2} = -\frac{Ac'}{c \cdot \cos^3\theta} \, . \qquad \text{[Eq. 4]}$$

Now for the radius of curvature ρ we have

$$\frac{1}{\rho} = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \left|\frac{c' \cos\theta}{c}\right| \, . \qquad \text{[Eq. 5]}$$

From

$$ds = \rho\,|d\theta| \qquad \text{[Eq. 6a]}$$

and the sign convention of Fig. 1 we arrive at

$$\left|\frac{ds}{d\theta}\right| = \left|\frac{c}{c' \cos \theta}\right| \qquad \text{[Eq. 6b]}$$

$$\frac{dx}{d\theta} = -\frac{c}{c'} \qquad \text{[Eq. 6c]}$$

$$\frac{dy}{d\theta} = -\frac{c}{c'} \tan \theta \qquad \text{[Eq. 6d]}$$

THE INTENSITY CALCULATION

The intensity calculation [Fig. 2] is based on the assumption that the energy is confined in an infinitesimally narrow bundle of rays. This implies that the intensity along a ray can be expressed in terms of that one particular ray.

$$I(x) = p \frac{d\Omega}{dF} = p \frac{2\pi \cos \theta_0 \, d\theta_0}{2\pi x \, dn} \qquad \text{[Eq. 7]}$$

where Ω is the unit solid angle and F the area swept out by the wave surface normal to the rays [Ref. 1]. (In the following, the emission of the point source p will assumed to be 1). It is possible to express dn in different ways, depending on which variables are looked upon as the independent ones. For a reason that will become clear later we will assume θ_0 and y to be independent. In the appendix we also give a variant with θ_0 and θ being independent.

A. The First Formula

From Fig. 2 it is seen that

$$dn = -\sin \theta \, dx = -\sin \theta \frac{\partial x}{\partial \theta_0} \, d\theta_0 \qquad \text{[Eq. 8]}$$

except, exactly at the turning point (x_H, y_H) where Eq. 8 is not valid;

after the turning point both dx and $\sin\theta$ change sign, abruptly. So, Eq. 7 reduces to,

$$I(x) = \frac{-\cos\theta_0}{x\sin\theta\,\dfrac{\partial x}{\partial\theta_0}} \, . \qquad \text{[Eq. 9]}$$

From Eq. 3 it follows that

$$x = \int_{y_0}^{y} \cot an\,\theta\, dy \, , \qquad \text{[Eq. 10]}$$

only valid if $x < x_H$, and from this

$$\frac{\partial x}{\partial\theta_0} = \int_{y_0}^{y} -\frac{1}{\sin^2\theta}\frac{\partial\theta}{\partial\theta_0}\, dy \quad x < x_H \qquad \text{[Eq. 11]}$$

From Snell's law [Eq. 2a] we know that

$$c_0\cos\theta = c\cos\theta_0 \, . \qquad \text{[Eq. 12]}$$

Partial differentiation with respect to θ_0 provides

$$\frac{\partial\theta}{\partial\theta_0} = \frac{\cos\theta\sin\theta_0}{\sin\theta\cos\theta_0} \, . \qquad \text{[Eq. 13]}$$

(We see the advantage of taking θ_0 and y as independent variables here, $\frac{\partial c}{\partial\theta_0}$ being zero.)

Substitution in Eq. 11 yields,

$$\frac{\partial x}{\partial\theta_0} = -\tan\theta_0\int_{y_0}^{y}\frac{\cos\theta}{\sin^3\theta}\, dy \quad x < x_H \qquad \text{[Eq. 14]}$$

With the aid of Eq. 3 this can be expressed as

$$\frac{\partial x}{\partial \theta_0} = -\tan \theta_0 \int_{x_0}^{x} \frac{1}{\sin^2 \theta} \, dx \qquad x < x_H \qquad \text{[Eq. 15]}$$

Substitution of this result in Eq. 9, provides,

$$I(x) = \frac{\cos^2 \theta_0}{x \sin \theta_0 \sin \theta \displaystyle\int_{x_0}^{x} \frac{1}{\sin^2 \theta} \, dx} \, . \qquad x < x_H \qquad \text{[Eq. 16]}$$

B. <u>Consideration of the First Formula</u>

There are some interesting points to make now with regard to this formula [Eq. 16].

Firstly, it appears to be the continuous form of the formula often used in the case of the piecewise linear sound velocity profile, viz.,

$$I(x) = \frac{\cos^2 \theta_0}{x \sin \theta_0 \sin \theta_n \displaystyle\sum_{i=1}^{n} \frac{x_i - x_{i-1}}{\sin \theta_{i-1} \sin \theta_i}} \, . \qquad \text{[Eq. 17]}$$

where i indicates the layer number.

Secondly, Eq. 16 is not valid after a turning point $(\theta = 0)$, is passed. The point $\theta = 0$ is a singular point of the integral and also the point where assumption [Eq. 8] does not hold. But the whole expression converges if we approach the turning point from the left side, as is easily proved in the case of a linear sound velocity profile. For, in this case the solution is

$$x = \frac{c_0}{g \cos \theta_0} (\sin \theta_0 - \sin \theta) \qquad \text{[Eq. 18a]}$$

$$dx = \frac{-c_0}{g \cos \theta_0} \, d \sin \theta \qquad \text{[Eq. 18b]}$$

(where g is the sound velocity gradient), and substitution in Eq. 16 provides

$$I(x) = \frac{\cos^2 \theta_0}{x^2} . \qquad \text{[Eq. 19]}$$

This is the solution for a linear profile shown to be true also after a turning point [Ref. 1]. Let us formally extend Eq. 16 for $x > x_H$, then for this example, after substitution of Eq. 18b

$$\sin\theta \int_{x_0}^{x} \frac{1}{\sin^2\theta}\, dx = -\frac{c_0 \sin\theta}{g \cos\theta_0} \left[\lim_{\theta_H \downarrow 0} \int_{\theta_0}^{\theta_H} \frac{d \sin\theta}{\sin^2\theta} + \lim_{\theta_H \uparrow 2\pi} \int_{\theta_H}^{\theta} \frac{d\sin\theta}{\sin^2\theta} \right] =$$

$$= \frac{c_0 \sin\theta}{g\cos\theta_0} \left[\lim_{\theta_H \downarrow 0} \frac{1}{\sin\theta_H} - \lim_{\theta_H \uparrow 2\pi} \frac{1}{\sin\theta_H} - \frac{1}{\sin\theta_0} + \frac{1}{\sin\theta} \right] \qquad \text{[Eq. 20]}$$

The integral from x_0 to x has been divided into one from x_0 to x_H and another from x_H to x.

We see that the first two terms in Eq. 20 lead to $+\infty$ at both limits and in order now to satisfy Eq. 19 they have to be ignored. They appear because we pass the turning point, that point where our initial assumption [Eq. 8] did not hold. Thirdly, from a numerical point of view [Eq. 16] is no longer usable in close proximity of the turning point, because $\sin\theta$ goes to zero and the integral becomes infinite.

C. <u>The Second Formula</u>

By means of partial integration we can transform the last two terms of the denominator in Eq.16, viz.,

$$\sin\theta \int_{x_0}^{x} \frac{1}{\sin^2\theta}\, dx = -\sin\theta \int_{x_0}^{x} \frac{dx}{d\theta}\, d \operatorname{cotan}\theta =$$

$$= \sin\theta \left\{ \frac{c}{c'} \operatorname{cotan}\theta \Big|_{x_0}^{x} - \int_{x_0}^{x} \operatorname{cotan}\theta \, d\, \frac{c}{c'} \right\} \qquad \text{[Eq. 21a]}$$

which may be developed further, under the condition that a second derivative of the sound velocity exists, to

$$\frac{c}{c'}\cos\theta - \sin\theta \operatorname{cotan}\theta_0 \frac{c_0}{c'_0} - \sin\theta \int_{x_0}^{x} \frac{c'^2 - cc''}{c'^2}\,dx \qquad \text{[Eq. 21b]}$$

where c_0 denotes the sound-velocity at the source. Substitution in Eq. 16, provides

$$I(x) = \frac{\cos^2\theta_0}{x\sin\theta_0\left[\frac{c}{c'}\cos\theta - \sin\theta\operatorname{cotan}\theta_0\frac{c_0}{c'_0} - \sin\theta\int_{x_0}^{x}\frac{c'^2 - cc''}{c'^2}\,dx\right]} \qquad \text{[Eq. 22]}$$

We can derive [Eq. 22] in a different way (see Appendix), from which it appears that this expression is also valid beyond the turning point. It is shown there that that derivation is in fact the more natural way of deriving [Eq. 22], and is totally independent of the turning point. But by a formal application of partial integration to Eq. 16 in a manner similar to Eq. 20 for $x > x_H$ we would obtain two extra terms in the denominator, viz.

$$\lim_{\theta_H\;0}\frac{c}{c'}\operatorname{cotan}\theta_H - \lim_{\theta_H\;2\pi}\frac{c}{c'}\operatorname{cotan}\theta_H \qquad \text{[Eq. 23]}$$

which both go to $+\infty$.

These two terms again express explicitly the wrong contribution due to the fact that our original assumption [Eq. 8] does not hold at the turning point, and the terms have to be ignored. It is easy to see that Eq. 22 is applicable in the proximity of the turning point; the last two terms in the denominator go to zero, but the first term remains finite.

D. Extension of First Formula beyond the Turning Point

From the fact that Eq. 22 is also valid after the turning point and the fact that Eqs. 22 and 16 are identical before the turning point, it follows that we can extend expression 16 after the turning point by writing

$$I(x) = \frac{\cos^2\theta_0}{x \left| \sin\theta_0 \quad \sin\theta \displaystyle\int_{x_0}^{x} \frac{1}{\sin^2\theta}\, dx \right|} \qquad \text{[Eq. 24a]}$$

where the bar in the integral means that in the neighbourhood of x_H by definition

$$\int_{x_H-\varepsilon}^{x_H+\varepsilon} \frac{1}{\sin^2\theta}\, dx = \frac{c}{c'} \cot\text{an}\,\theta \Bigg|_{x_H-\varepsilon}^{x_H+\varepsilon} - \int_{x_H-\varepsilon}^{x_H+\varepsilon} \frac{c'^2 - cc''}{c'^2}\, dx \qquad \text{[Eq. 24b]}$$

where ε is some finite number >0 and $|c'|>0$ for $|x - x_H| < \varepsilon$.

We see that with this definition it happens that the integral of Eq. 24a changes sign after the turning point, and we get the shape as sketched in Fig. 3.

Numerical Example

Formulae 22 and 24 have been implemented in a computer program. The ray equations are converted to a set of first order equations and solved numerically, using a Kutta Merson procedure [Ref. 2].

The sound velocity profile is approximated by Cubic Spline interpolation [Ref. 3]. The advantage of this interpolation method is that a continuous second derivative of the sound velocity, is obtained. The advantage of the Kutta Merson method is that it is highly accurate, 4-th order, and that its step-size is automatically adapted to the required accuracy during the integration. With each integration step the intensity contribution is calculated either with Eqs. 22 or 24 depending on the magnitude of c' and $\sin\theta$. For instance, if Eq. 24 is used and $\sin\theta$ decreases below a certain limit, the integral of Eq. 22 is at that point calculated by equating both denominators of Eqs. 22 and 24 and the calculation is continued with Eq. 22 until $\sin\theta$ is large enough to continue the calculation with Eq. 24.

The example in Fig. 4 illustrates the method very well. There are two points where c' equals zero and, for all the rays, there is one point where $\sin\theta$ becomes zero. The intensity peaks correspond to the caustic. They indicate the points where the denominator is zero. We also see that for the first part all the intensity curves coincide. The results there can be compared with the solution for the linear profile, formula 19.

CONCLUSION

A method has been presented for the intensity calculation in the case of a sound velocity profile which is continuous up to the second derivative. The intensity is calculated along the ray simultaneously with the numerical integration of the ray differential equations. It appears that the intensity can be expressed by two different formulae, theoretically valid everywhere, but numerically one is not usable near the turning point, and the other is not usable near points where c' is zero. By switching from one formula to the other in the computer program these difficulties can be avoided.

REFERENCES

1. C.B. Officer, "Introduction to the Theory of Sound Transmission", McGraw-Hill, 1958

2. P.M. Lukehart, "Algorithm 218 - Kutta Merson", Collections Assoc. Comp. Mac., Vol. 4, p.273, 1966.

3. C.B. Moler and L.P. Solomon, "Use of Splines and Numerical Integration in Geometrical Acoustics", J. Acoust. Soc. Am., Vol. 48, No. 3 (Pt.2), 1970.

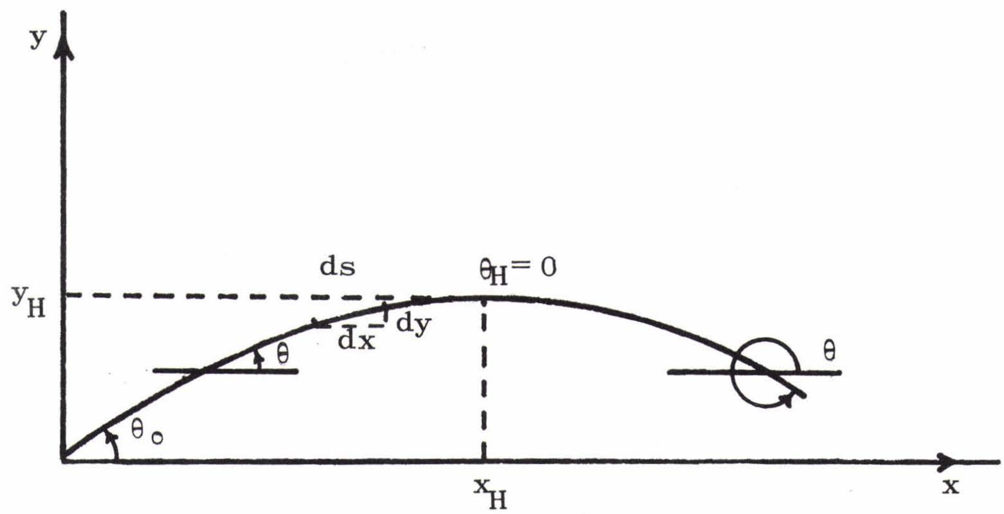


FIG. 1

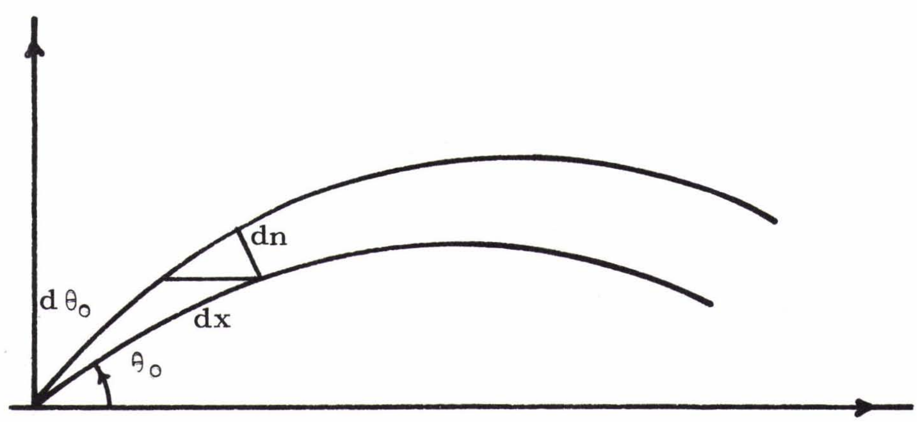
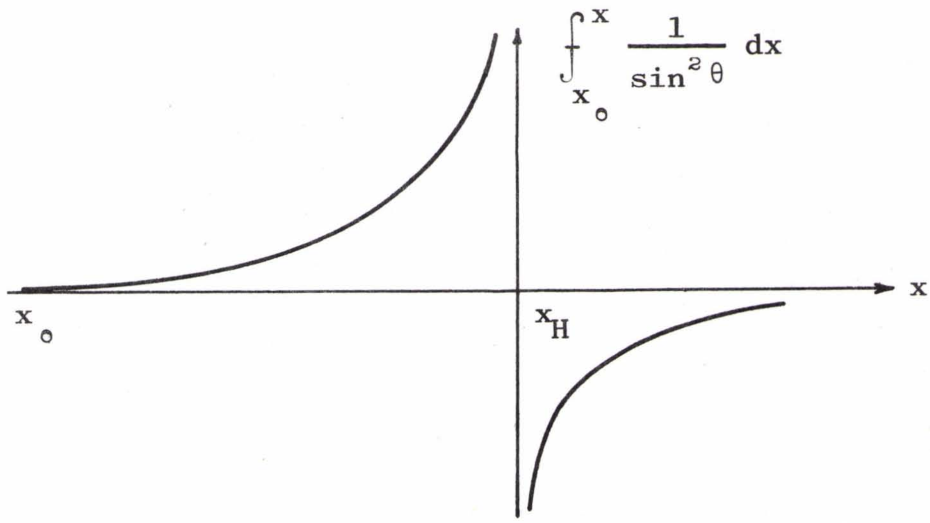
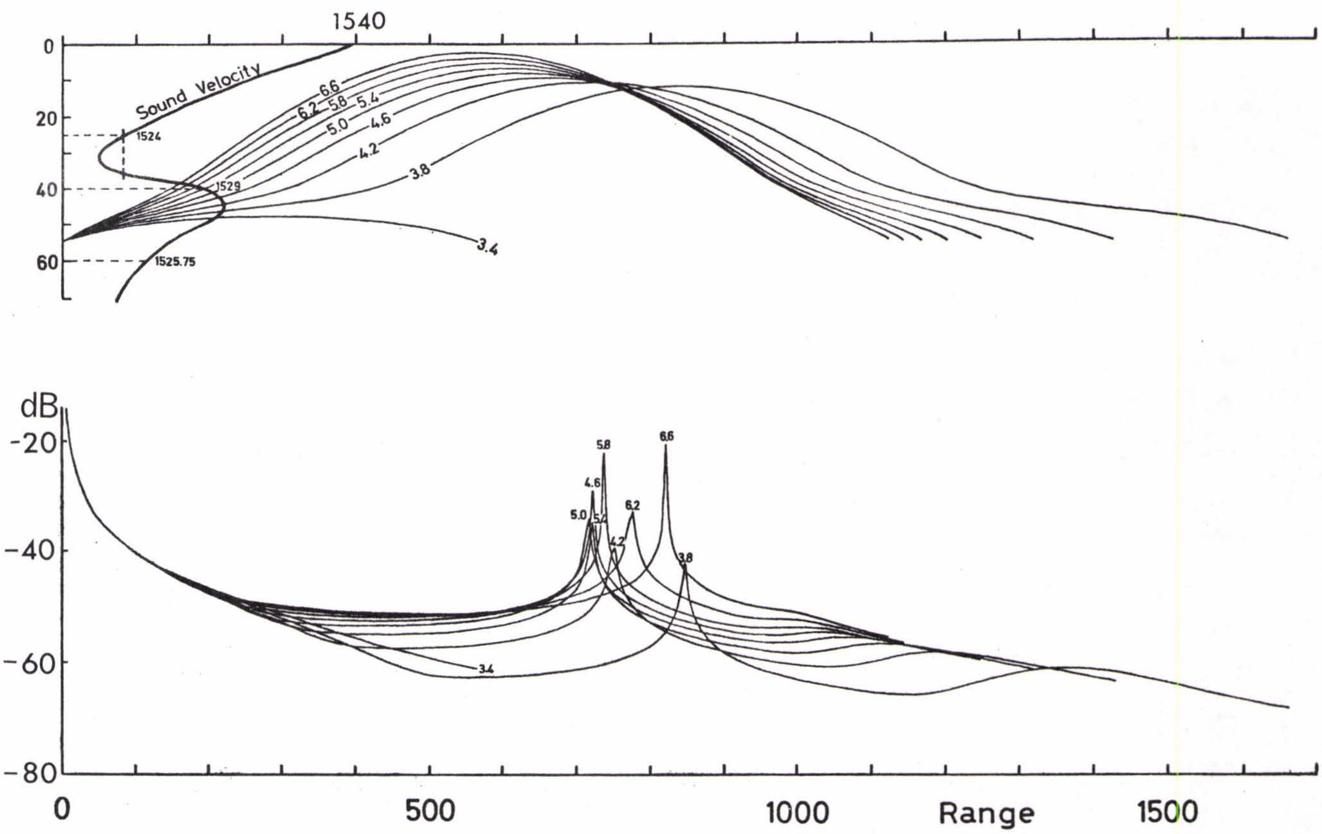


FIG. 2



$\int_{x_o}^{x} \frac{1}{\sin^2\theta}\,dx$
x
x_o
x_H

FIG. 3



1540
0
20
40
60
Sound Velocity
1524
1529
1525.75
6.6
6.2
5.8
5.4
5.0
4.6
4.2
3.8
3.4
dB
-20
-40
-60
-80
0
500
1000
Range
1500

FIG. 4

APPENDIX A

ANOTHER DERIVATION OF THE SECOND FORMULA

Equation 22 can be derived in a different way [see Fig. A.1]. At P, the point where the intensity is required, we transform coordinates

$$n = -x \sin \theta + y \cos \theta \qquad \text{[Eq. A.1a]}$$

$$t = x \cos \theta + y \sin \theta \qquad \text{[Eq. A.1b]}$$

We now choose θ_0 and θ as independent variables. Instead of Eq. 8 we develop dn as,

$$dn = \frac{\partial n}{\partial \theta_0} d\theta_0 . \qquad \text{[Eq. A.2]}$$

This leads to

$$I(x) = \frac{\cos \theta_0}{x \dfrac{\partial n}{\partial \theta_0}} \qquad \text{[Eq. A.3]}$$

From Eq. A.1a it follows that

$$\frac{\partial n}{\partial \theta_0} = -\sin \theta \frac{\partial x}{\partial \theta_0} + \cos \theta \frac{\partial y}{\partial \theta_0} . \qquad \text{[Eq. A.4]}$$

From Eq. 6c we know that

$$x = -\int_{\theta_0}^{\theta} \frac{c}{c'} d\theta . \qquad \text{[Eq. A.5]}$$

Partial differentiation with respect to θ_0 of Eqs. A.5 and 12 (Snell's law) respectively provides

$$\frac{\partial x}{\partial \theta_0} = -\int_{\theta_0}^{\theta} \frac{\partial\left[\frac{c}{c'}\right]}{\partial \theta_0} \, d\theta + \frac{c_0}{c'_0} \qquad \text{[Eq. A.6a]}$$

$$\frac{\partial y}{\partial \theta_0} = \frac{c}{c'} \tan \theta_0 \qquad \text{[Eq. A.6b]}$$

Because

$$\frac{\partial\left[\frac{c}{c'}\right]}{\partial \theta_0} = \frac{d\left[\frac{c}{c'}\right]}{dy} \frac{\partial y}{\partial \theta_0} = \frac{\partial\left[\frac{c}{c'}\right]}{\partial y} \frac{\partial y}{\partial \theta_0} \qquad \text{[Eq. A.7]}$$

it follows that

$$\frac{\partial x}{\partial \theta_0} = \frac{c_0}{c'_0} + \tan \theta_0 \int_{x_0}^{x} \frac{c'^2 - cc''}{c'^2} \, dx \; . \qquad \text{[Eq. A.8]}$$

Substitution of Eqs. A.6 and A.8 in Eq. A.4 and substitution of this result in Eq. A.3 gives Eq. 22. This derivation of Eq. 22 is the most natural, since it is independent of the turning point and, in addition, proves the extension of Eq. 16 to Eq. 24.

DISCUSSION

It was suggested that some of the problems of calculating intensity along a single ray might be avoided by the use of a second order differential equation and the consideration of energy propagation along the ray.

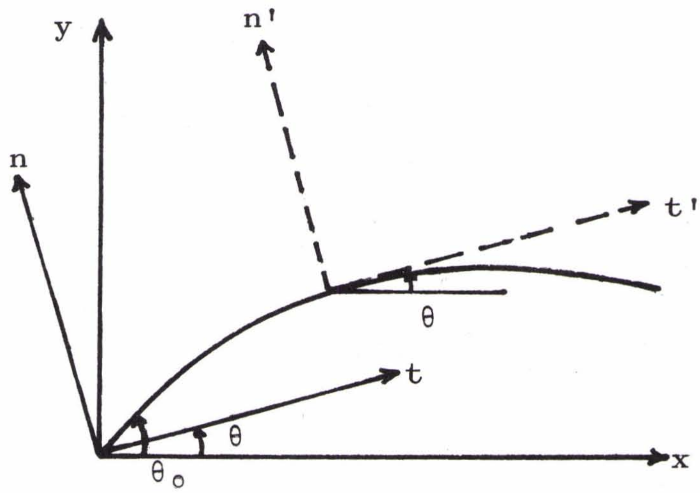


FIG. A.1