Shallow Water Reverberation Modeling Using An Autoregressive Range-Scattering Function Approach

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Abstract

The primary limitation on system performance for active sonars is reverberation — the reradiation of acoustic energy back to the receiver from inhomogeneities in the physical properties of the medium and its boundaries. A new technique for modeling reverberation using an autoregressive power spectral density function to describe the range-scattering function of the reverberation channel is presented. The method is used to estimate the range-scattering function from sampled reverberation time-series.

1. Introduction

Reverberation — the reradiation of acoustic energy back to the receiver from inhomogeneities in the physical properties of the medium and its boundaries can be particularly bothersome in shallow water when contributions from surface, volume, and bottom reverberation are present. Many techniques for modeling reverberation have been proposed, developed, and tested. These efforts have been motivated primarily by either the need to simulate reverberation time-series data to test existing processing techniques ([1], [2]) or the desire to formulate new detector-processing structures [3]. A new technique for modeling reverberation that can be used to address either of these two issues is presented below.

2. Linear Systems Theory Approach To Reverberation Modeling

Small amplitude acoustic signals are governed by the linear wave equation [4]; therefore, reverberation can be analyzed using a linear, time-varying filter. The input/output relationship of such a filter is

$$y(t) = \int_{-\infty}^{\infty} s(t-\tau)h(t;\tau)d\tau, \tag{1}$$

where $s(t)$ is the input signal, $y(t)$ is the output signal and $h(t;\tau)$ describes the output of the filter at time t due to a unit impulse applied τ seconds earlier. It is assumed that s, and therefore, y have negligible energy outside a band centered at some carrier frequency and can be designated as band-pass signals [3]. Analysis then continues with the equivalent lowpass complex representation of these signals and the impulse response h. To simplify notation, it is now assumed that s, h, and y refer to their complex lowpass representations.

Equation (1) will be equated to the system shown in Figure 1. Specifically, the deterministic signal $s(t)$ is the transmit waveform and $y(t)$ is the resultant reverberation time-series. The time-varying filter $h(t;\tau)$ models the effects of system parameters, propagation effects, and scattering effects and will be referred to as the "impulse response" of the reverberation channel. From this model it can be seen that τ is a variable representing time delay.

A simple approach to modeling the reverberation channel that has found widespread use assumes a first-order, uncorrelated scattering model [5]. It is assumed that the scattering regions are uncorrelated with each other no matter how closely spaced in the time-delay variable τ, and, at each time delay, the impulse response is a sample function of a stationary, zero-mean, complex Gaussian random process. These assumptions are typically referred to as the "wide-sense stationary, uncorrelated scattering (WSSUS) assumptions." Because of the zero-mean, Gaussian assumption, the reverberation channel is completely characterized by the second-order statistics of $h(t;\tau)$. As a result of the WSSUS assumptions, the correlation function of $h(t;\tau)$ has the unique structure

$$E[h(t_1;\tau_1)h^*(t_2;\tau_2)] = R_h(t_1 - t_2;\tau_1)\delta(\tau_1 - \tau_2). \tag{2}$$

That is, in the variable t, the correlation function is only a function of the time difference $\Delta t = t_1 - t_2$, and in the variable τ, the correlation function has the structure of a nonstationary, white-noise process.

The time variations of the reverberation channel are caused by the motion of the medium inhomogeneities and the motion of the transmit/receive array relative to the medium. As measured by the variable Δt above, this motion manifests itself as a Doppler spread of the transmitted waveform. Under certain conditions [6] that are assumed to hold in this development, the channel can be assumed to be time invariant so that (1) reduces to a simple convolution integral

$$y(t) = \int_{-\infty}^{\infty} s(t-\tau)h(\tau)d\tau, \tag{3}$$

and the correlation function in (2) reduces to

$$\begin{aligned} E[h(\tau_1)h^*(\tau_2)] &= E[|h(\tau_1)|^2]\delta(\tau_1 - \tau_2) \\ &= r_h(\tau_1)\delta(\tau_1 - \tau_2). \end{aligned} \tag{4}$$

Now the impulse response of the reverberation channel is just a realization of a nonstationary, complex Gaussian, white-noise process where the average intensity function $r_h(\tau)$ is referred to as the "range-scattering function" [3]. Although $r_h(\tau)$ is a causal function of time delay and, in theory, is of infinite extent, for all practical purposes, there is a range of values of τ over which $r_h(\tau)$ is essentially nonzero. This limit will be denoted as τ_h. It is often referred to as the "multipath spread of the channel" [7].

The continuous-time Fourier Transform (CTFT) of $h(\tau)$ is also a stochastic process with correlation function,

$$\begin{aligned} R_H(f_1, f_2) &= E[H(f_1)H^*(f_2)] \\ &= E\left[\int_{-\infty}^{\infty} h(\tau_1)\exp(-j2\pi f_1\tau_1)d\tau_1 \int_{-\infty}^{\infty} h^*(\tau_2)\exp(j2\pi f_2\tau_2)d\tau_2\right] \\ &= \int_{-\infty}^{\infty} r_h(\tau)e^{j2\pi\tau(f_2-f_1)}d\tau \end{aligned} \tag{5}$$

that is only a function of the frequency difference $\Delta f = f_2 - f_1$ and therefore can be written as

$$R_H(\Delta f) = \int_{-\infty}^{\infty} r_h(\tau)e^{j2\pi\Delta f\tau}d\tau. \tag{6}$$

Therefore $H(f)$, the random transfer function of the reverberation channel, is a wide-sense stationary (WSS), zero-mean, complex Gaussian process. Also note that, from the Wiener-Khinchen theorem ,

$$r_h(\tau) = \int_{-\infty}^{\infty} R_H(\Delta f)e^{-j2\pi\Delta f\tau}d\Delta f. \tag{7}$$

That is, the range-scattering function is the CTFT of the correlation function $R_H(\Delta f)$. $R_H(\Delta f)$ is referred to as the "two-frequency correlation function" [3]. Since $R_H(\Delta f)$ is an autocorrelation function in the frequency variable Δf, it yields a measure of the frequency coherence of the channel. If f_h denotes a measure of the coherence bandwidth, then $f_h \approx \frac{1}{\tau_h}$. Two sinusoids with frequency separation greater than f_h are affected differently by the channel. It can also be shown that the time-varying intensity of the reverberation is

$$R_y(t,t) = E\left[|s(t-\tau)|^2 r_h(\tau)d\tau\right] = \int_{-\infty}^{\infty} |s(t-\tau)|^2 r_h(\tau)d\tau. \tag{8}$$

In summary, a range spread reverberation channel has been modeled as a random, time-invariant, linear system with a WSS, zero-mean complex Gaussian transfer function $H(f)$. The correlation function of the transfer function is the two-frequency correlation function in (6) and its power spectral density (PSD) function is the range-scattering function in (7). Knowledge of either of these second-order statistics completely characterizes the reverberation.

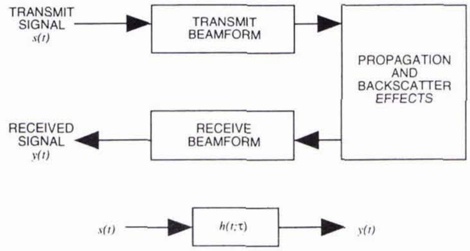

Figure 1: Linear System Model for Reverberation

3. Discrete Processes

The ultimate goal is to estimate the range-scattering function $r_h(\tau)$ from reverberation time-series data, and to this end the transfer function $H(f)$ is now sampled at equally spaced intervals δf along the frequency axis. The motivation being that (6) and (7) are equivalent to a PSD estimation problem in the time delay (τ) and frequency (Δf) domains. For the present time, it is assumed that access to the samples is available. Later it will be shown how to extract this information from the reverberation time-series data. The discrete-time process

$$H[n] = H(f = n\delta f),\ n = \cdots, -1, 0, 1, \cdots \tag{9}$$

is formed. $H[n]$ is a discrete-time, zero-mean, complex Gaussian WSS process with autocorrelation function $R[m] = R_H(\Delta f = m\delta f)$. As such, the z-transform of $R[m]$ can be defined as

$$r(z) = \sum_{m=-\infty}^{\infty} R[m] z^{-m}. \tag{10}$$

When z is limited to the unit circle, $r(z)$ is the discrete time Fourier transform (DTFT) of $R[m]$. In addition, if z is expressed as a continuous function of τ with δf held fixed, then on the unit circle, $z(\tau) = e^{j2\pi\tau\delta f}$ and

$$r(\tau) = r(z(\tau)) = r(e^{j2\pi\tau\delta f}) = \sum_{m=-\infty}^{\infty} R[m] e^{-j2\pi m\tau\delta f}, \tag{11}$$

and it is seen that $r(\tau)$, the DTFT of $R[m]$, is the PSD of the WSS process $H[n]$. Note that $r(\tau)$ is periodic in τ with period $= 1/\delta f$.

Since $R[m]$ was derived by sampling the continuous function $R_H(\Delta f)$ at equally spaced samples δf along the frequency axis, the sampling theorem yields,

$$r(\tau) = \frac{1}{\delta f} \sum_{k=-\infty}^{\infty} r_h(\tau - \frac{k}{\delta f}); \tag{12}$$

that is, the PSD of the WSS process $H[n]$ is equal to a periodic function of τ that is a scaled, infinite sum of displaced replicas of the scattering function $r_h(\tau)$. As long as the sampling interval in the frequency domain satisfies

$$\delta f \leq \frac{1}{\tau_h}, \tag{13}$$

aliasing in the time-delay domain is kept to a minimum and $r(\tau)$ is a good approximation (to within a scale factor) of $r_h(\tau)$. Note that in order to minimize aliasing, the frequency samples must be spaced less than the coherence bandwidth of the channel ($\delta f \leq f_h$).

3.1. Autoregressive Processes

It is now assumed that only a finite number of samples ($H[n]$, $n = 0, 1, \ldots, N-1$) are available and that $r(\tau)$, which was previously shown to be a good approximation to the scattering function $r_h(\tau)$, is desired. The problem now falls into the category of PSD estimation – determining the spectral content of a random process based on a finite set of observations from that process. Typically, the set of observations is made in the time domain and the

PSD estimate generated in the frequency domain, so *it is important to remember that in the current problem, the observations are made in the frequency domain, and the spectral estimates are generated in the time-delay domain.*

The approach taken in this effort assumes that $H[n]$, $n = 0, 1, \ldots, N-1$ are samples of a zero-mean, complex Gaussian, autoregressive process of known order p (referred to as an "AR(p) process") [8]

$$H[k] = -\sum_{n=1}^{p} a[n]H[k-n] + u[k]. \tag{14}$$

Here p is the order of the autoregressive process, $\mathbf{a} = [a[1]\, a[2] \cdots a[p]]^T$ is a vector whose elements are the complex AR coefficients, and $u[k]$ is a complex Gaussian, white-noise process with variance σ^2. The PSD of $H[k]$ is

$$r(\tau') = \frac{\sigma^2}{\left|1 + \sum_{m=1}^{p} a[m] \exp(-j2\pi m \tau')\right|^2} \tag{15}$$

where τ' represents a normalized time delay

$$\tau' = \frac{\tau}{\tau_h}. \tag{16}$$

This result is consistent with the requirement for minimizing aliasing in the time-delay domain, and the fundamental period of τ' is chosen to be $0 \leq \tau' \leq 1$.

4. Maximum Likelihood Estimation Of The Range-Scattering Function

In this section maximum likelihood estimates (MLEs) of $\mathbf{a}$ and σ^2 denoted as $\hat{\mathbf{a}}$ and $\hat{\sigma}^2$ are developed [9]. When these estimates are substituted into (15), a maximum likelihood estimate of the PSD (range-scattering function) is produced:

$$\hat{r}(\tau') = \frac{\hat{\sigma}^2}{\left|1 + \sum_{m=1}^{p} \hat{a}[m] \exp(-j2\pi m \tau')\right|^2}. \tag{17}$$

Recall that the MLEs are based on the data set $H[n]$, $n = 0, 1, \ldots, N-1$, which is denoted in vector form as $\mathbf{H}$. The approximate (actually conditional) probability density function (PDF) of $\mathbf{H}$ is:

$$p(\mathbf{H}; \mathbf{\Theta}) = \frac{1}{(\pi\sigma^2)^{N'}} \exp\left(-\frac{1}{\sigma^2} \sum_{n=p}^{N-1} \left|\sum_{k=0}^{p} a[k]H[n-k]\right|^2\right), \tag{18}$$

where the vector of unknown parameters $\mathbf{\Theta} = [\mathbf{a}^T \sigma^2]^T$, $N' = N - p$ and $a[0] = 1$. Maximizing this function with respect to σ^2 yields

$$\sigma^2 = \frac{1}{N'} J(\mathbf{a}) = \frac{1}{N'} \sum_{n=p}^{N-1} \left|\sum_{k=0}^{p} a[k]H[n-k]\right|^2. \tag{19}$$

When $\mathbf{a}$ is replaced by its MLE, σ^2 becomes $\hat{\sigma}^2$.

The MLE of $\mathbf{a}$ is found by minimizing $J(\mathbf{a})$ in (19). Note that $J(\mathbf{a})$ is a quadratic with respect to the $a[k]$'s and can be written as

$$J(\mathbf{a}) = \|\mathbf{H}_p + \mathbf{G}\mathbf{a}\|^2, \tag{20}$$

where

$$\mathbf{H}_p = [H[p]\, H[p+1] \cdots H[N-1]]^T, \tag{21}$$

and

$$\mathbf{G} = \begin{bmatrix} H[p-1] & H[p-2] & \ldots & H[0] \\ H[p] & H[p-1] & \ldots & H[1] \\ \vdots & \vdots & \ddots & \vdots \\ H[N-2] & H[N-3] & \ldots & H[N-1-p] \end{bmatrix}. \tag{22}$$

Minimizing $J(\mathbf{a})$ with respect to $\mathbf{a}$ yields

$$\hat{\mathbf{a}} = -\left(\mathbf{G}^H\mathbf{G}\right)^{-1}\mathbf{G}^H\mathbf{H}_p, \tag{23}$$

and substituting (23) into (19) yields

$$\hat{\sigma}^2 = \frac{1}{N'}\mathbf{H}_p^H\mathbf{P}_G^{\perp}\mathbf{H}_p, \tag{24}$$

where $\mathbf{P}_G^{\perp}$ is the projection matrix

$$\mathbf{P}_G^{\perp} = \mathbf{I} - \mathbf{G}\left(\mathbf{G}^H\mathbf{G}\right)^{-1}\mathbf{G}^H. \tag{25}$$

5. Processing Reverberation Time-Series

The previous development assumed that samples from the random process $H[n]$ were readily available. In reality, a finite set of samples must be derived from the time-series $y(t)$. These can be obtained as follows. Recall that $s(t)$ is a complex lowpass signal and $y(t)$, $h(t)$ are complex lowpass random processes. Assume that the energy of $s(t)$ is negligible outside the band $-B/2 \leq f \leq B/2$. Also assume that the bandwidth of the signal is greater than the coherence bandwidth of the channel, $B > f_h \approx 1/\tau_h$. Sample $y(t)$ with sample period $\Delta t = \frac{1}{B}$, and form the discrete-time version of (3)

$$y[n] = \sum_{k=0}^{N_h-1} h[k]s[n-k] \quad n = 0, 1, \ldots, N_T - 2, \tag{26}$$

where $s[n] = s(t = n\Delta t)$, $h[n] = h(t = n\Delta t)$, $y[n] = y(t = n\Delta t)$, $N_T = N_h + N_s$, $N_h = \frac{\tau_h}{\Delta t}$, $N_s = \frac{T_s}{\Delta t}$ and T_s is the duration of the transmit waveform. The discrete-time system replaces the continuous-time system with negligible loss of information. Equation (26) represents a tapped delay line model of the reverberation channel [3].

Now assume that $\tau_h \gg T_s$ so that $N_h \gg N_s$ and assume that $N \approx N_T \approx N_h$ samples of $y[n]$ are collected. Performing an N point discrete Fourier transform (DFT) on $y[n]$, the linear convolution in (26) can be approximated with a circular convolution and can be written in the frequency domain as

$$\tilde{Y}[k] = \tilde{H}[k]\tilde{S}[k], \; k = 0, 1, \ldots, N-1, \tag{27}$$

where the DFT of $y[n]$ is defined to be $\tilde{Y}[k] = \sum_{n=0}^{N-1} y[n]\exp(-j2\pi nk/N)$, $k = 0, 1, \ldots, N-1$.
Now $\tilde{H}[k]$ is derived from $\tilde{Y}[k]$ by division under the assumption that $\tilde{S}[k] \neq 0$, $k = 0, 1, \ldots, N-1$.

Given the sample period above, the DFT approximates the CTFT reasonably well, that is, $\tilde{Y}[k]$ can be replaced with $Y[k]$, $\tilde{S}[k]$ with $S[k]$, and $\tilde{H}[k]$ with $Y[k]$. Also note that the discrete frequency index k corresponds to frequencies $\frac{k}{N\Delta t} \approx \frac{k}{\tau_h + T_s}$ so the sample period of the DFT output is $\frac{1}{\tau_h + T_s} < \frac{1}{\tau_h}$, which satisfies the requirement to minimize aliasing of the scattering function in (13). Note also that the $h[n]$ are samples from an independent, complex Gaussian random process, that is $h[n] \sim \mathcal{CN}(0, r[n])$, where

$$r[n] = r_h(\tau = n\Delta t). \tag{28}$$

The tap weights of the tapped delay line model are independent, zero-mean, complex Gaussian random variables, where the variance of $h[n]$ is $r[n]$. Finally, from (12), (15), (16), and (28), it can be seen that

$$\begin{aligned} r[n] &= r_h(\tau = n\Delta t) = r(\tau = n\Delta t) \\ &= r(\tau' = \frac{n\Delta t}{\tau_h}) = r(\tau' = \frac{n}{N_h}) \approx r(\tau' = \frac{n}{N}), \\ & \quad n = 0, 1, \ldots, N-1. \end{aligned} \tag{29}$$

The $r[n]$ are referred to as the "sampled range-scattering function" and are the quantities that are estimated with the MLE technique described in the previous section. In order to illustrate the efficacy of this approach and point out some of the processing details, several examples using actual in-water data are presented in the next section.

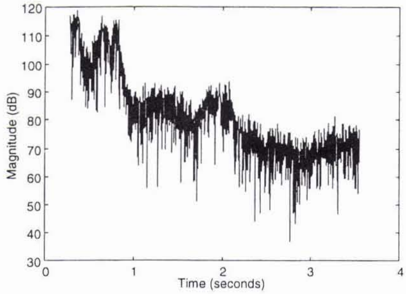

Figure 2: Time-series: Ping 1

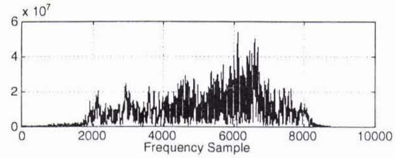

Figure 3: Spectral Content of Ping 1

6. IN-WATER RESULTS

The data were collected in approximately 100 meters of water. The sonar array transmitted a large time-bandwidth waveform ($T_s \approx 0.3$ seconds, $B \approx 2000$ Hz) and received the reverberation return while operating at a depth of approximately 30 meters. The element level data of the sonar array were processed to produce a digital version of the complex lowpass representation of the reverberation returns and the sum beam time-series was generated from the element level outputs. This is the time-series referred to as "$y[n]$" in (26). Figure 2 is the reverberation return time-series from the data set designated "Ping 1."

Figure 3 shows the spectral content of the time-series data of Ping 1 obtained by performing a DFT on the data. Note that this time-series possesses good spectral support over a wide frequency range (i. e. , $S[k] \neq 0$ over a wide range of frequencies). Since the AR modeling approach requires the estimation of only a small number of parameters ($p + 1$, and in the examples shown in this section, $p = 13$ was chosen), not all of the data in Figure 3 must be processed. Frequency samples $5000 - 6000$ were chosen for this example and the deconvolution operation was performed.

When these data are inserted into (17), the MLEs of the AR coefficients produce an MLE of the sampled range-scattering function shown in Figure 4. When this scattering function estimate is convolved with the squared magnitude of the transmit signal (essentially a boxcar function of duration T_s seconds), an estimate of the intensity of the received time-series is produced (see (8) for the continuous-time version of this relationship). Figure 5 shows the intensity estimate for Ping 1. Note that there are transient effects contained in the results, but when these effects are removed and the intensity estimate is plotted along with the original time-series data, as in Figure 6, the intensity estimate follows the complex structure of the reverberation time-series very closely.

The scattering function estimate (Figure 4) was then used to generate a realization of the random impulse response via the use of a random number generator. This impulse response was used to simulate a reverberation time-series via (26). Figure 7 shows a plot of the original time-series in the top trace and the simulated time-series in the bottom trace. The similarity between the two time-series appears to be rather close, indicating that AR modeling of the range-scattering function shows promise. Note that additional simulated realizations of the time-series can be obtained by simply generating new realizations of the impulse response using the estimate of the sampled range-scattering function and the random number generator.

Another data set was examined to test the robustness of the AR approach. Data set designated "Ping 2" was processed in the same manner as Ping 1. This includes using the same model order $p = 13$. Since the sonar was moving through the water, these data were taken in slightly different water mass from Ping 92. Figure 8 shows the intensity function estimate and the in-water time-series for Ping 2. Comparing Figures 6 and 8, it is readily seen that each of the time-series is quite different, but the AR approach is able to capture the complex structure of each.

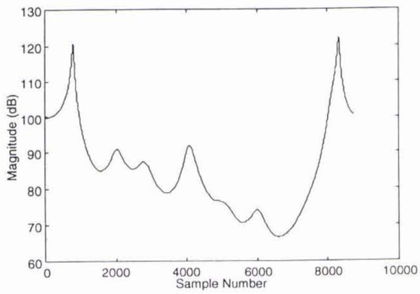

Figure 4: Scattering Function Estimate: Ping 1

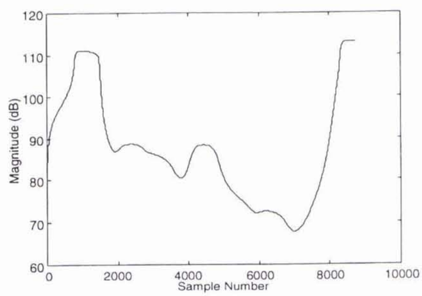

Figure 5: Intensity Function Estimate: Ping 1

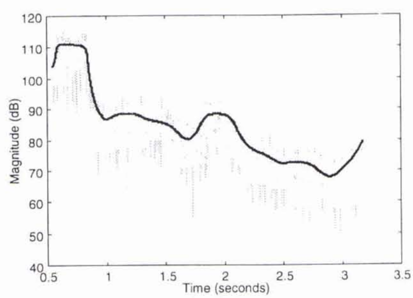

Figure 6: Intensity Function Estimate and Time-series: Ping 1

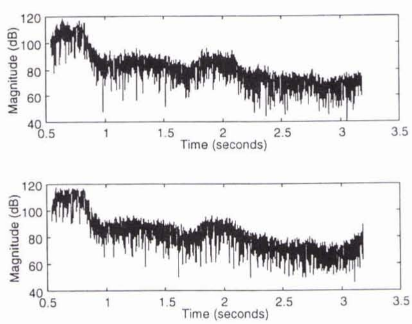

Figure 7: Actual and Simulated Time-series: Ping 1

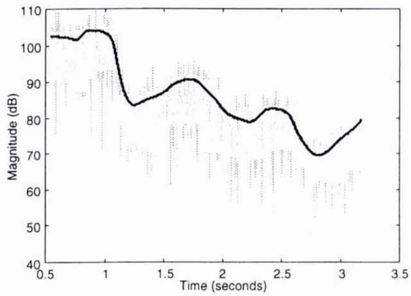

Figure 8: Intensity Function Estimate and Time-series: Ping 2

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