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by

LEONARD FORTUIN

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J.G. Retallack
J.G. Retallack
Director

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The wave equation in a medium with a time-dependent boundary

Leonard Fortuin

SACLANT ASW Research Centre, La Spezia, Italy

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It is shown that the Helmholtz equation is not exactly correct for a medium with a time-dependent boundary. The equation can be used with very good approximation when the time-derivative of the surface elevation is much smaller than the speed of the waves through the medium. For underwater sound waves, reflected and scattered by an ocean surface that can be described by the Pierson–Moskowitz spectrum, this means that the wind speed has to be much less than the sound speed.

Subject Classification: 13.2, 13.4.

LIST OF SYMBOLS

A	amplitude
c_0	sound speed
h	standard deviation of surface elevation
i	$(-1)^{\frac{1}{2}}$
k	wavenumber of radiation
$\tilde{p}$	pressure (time dependent)
p	pressure (time independent)
p_b	pressure, due to the boundary
p_0	pressure in unbounded medium
$\mathbf{R}$	vector in horizontal plane
t	time
v	wind speed
γ_T, γ_R	direction cosines
ζ	surface profile
ρ_0	density
τ	time difference
Φ	correlation function of surface elevation
ω	angular frequency of incident wave
ω_s	frequency of surface wave

INTRODUCTION

The propagation of sound waves through the ocean is governed by a wave equation. In case of a monochromatic source this wave equation is usually written as[1]

$$(\nabla^2+k^2)p=0, \tag{1}$$

the so-called Helmholtz equation. This equation is exact when the boundaries of the medium in which the waves are propagating are independent of time. However, in case of a time-dependent boundary, such as the ocean surface, the Helmholtz equation can only be used when the boundary changes slowly enough. It is the purpose of this paper to derive a condition for the validity of Eq. 1 in that case. Also, the significance of the derived condition will be discussed.

I. THE CONDITION FOR THE VALIDITY OF THE HELMHOLTZ EQUATION

It is convenient to deal with the velocity potential $\tilde{U}$ rather than the sound pressure $\tilde{p}$. For sound waves of small amplitude these two quantities are related by the equation

$$\tilde{p}=\rho_0\partial\tilde{U}/\partial t. \tag{2}$$

The velocity potential satisfies the equation[2]

$$\nabla^2\tilde{U}=c_0^{-2}\partial^2\tilde{U}/\partial t^2. \tag{3}$$

Before treating the case of a time-dependent surface, we consider the "frozen" boundary for comparison.

A. The Time-Independent Surface

If the sound source is monochromatic, $\tilde{U}$ can be written as $\tilde{U}(t)=U\exp(i\omega t)$, where U is independent of time. We have then

$$\partial\tilde{U}/\partial t=i\omega\tilde{U}, \tag{4}$$

$$\partial^2\tilde{U}/\partial t^2=-\omega^2\tilde{U}, \tag{5}$$

and

$$\tilde{p}=i\omega\rho_0\tilde{U}, \tag{6}$$

so that Eq. 3 changes into

$$(\nabla^2+k^2)\tilde{U}=0. \tag{7}$$

Using Eq. 6 and dropping the time-factor $\exp(i\omega t)$, Eq. 1 is found without any approximation.

B. The Time-Variant Surface

The movement of the surface causes the sound field to depend on time in a more complicated way than in the foregoing case. To illustrate this we refer to the Doppler effect that is present in the scattered field, hence also in the total field $\tilde{p}$ and so in $\tilde{U}$. For this reason U is now a function of time.

Consequently, instead of Eqs. 4 and 5 we find:

$$\partial\tilde{U}/\partial t=(i\omega U+\partial U/\partial t)\exp(i\omega t), \tag{8}$$

$$\partial^2\tilde{U}/\partial t^2=(-\omega^2U+i2\omega\partial U/\partial t+\partial^2U/\partial t^2)\exp(i\omega t), \tag{9}$$

and this result substituted into Eq. 3 does not produce Eq. 7. However, if the inequality

$$|\partial U/\partial t|\ll|\omega U| \tag{10}$$

is satisfied, Eqs. 8 and 9 may be replaced by Eqs. 4 and 5, and Eq. 7 does follow. So this is a sufficient condition for the validity of the Helmholtz equation in the case of a time-dependent boundary. In that case Eq. 6 also holds, so that the condition can be written as

$$|\partial p/\partial t|\ll|\omega p|. \tag{11}$$

It should be noted that p is the total sound field: $p=p_0+p_b$. As the incident field p_0 is independent of time, we get

$$|\partial p_b/\partial t|\ll\omega|p_0+p_b|, \tag{12}$$

where $\omega>0$.

If either $|p_0|\ll|p_b|$ or $|p_0|\gg|p_b|$, this condition is certainly satisfied when

$$|\partial p_b/\partial t|\ll\omega|p_b|. \tag{13}$$

But if $|p_0|$ and $|p_b|$ are of the same order of magnitude (as occurs close to the boundary), we have to observe the complete condition (Eq. 12) instead of the reduced condition (Eq. 13). The significance of both conditions is discussed in Sec. II.

II. SIGNIFICANCE OF THE CONDITIONS

A. The Reduced Condition

In Ref. 1 (Eq. 39) a formula is given for the scattered field that is derived from the Helmholtz equation. The time dependency is concentrated in the factor

$$F(t)\equiv\exp\{ik[\gamma_T\zeta(\mathbf{R}_1,t_1)+\gamma_R\zeta(\mathbf{R}_0,t_0)]\}, \tag{14}$$

where γ_T and γ_R are constants, smaller than or equal to 1. Replacing p_b by F in Eq. 13, we find the inequality

$$|\gamma_T\partial\zeta_1/\partial t_1+\gamma_R\partial\zeta_0/\partial t_0|\ll c_0. \tag{15}$$

This formula may be replaced by[3]

$$2|\partial\zeta/\partial t|\ll c_0, \tag{16}$$

as ζ is a stationary process and γ_T, $\gamma_R\leq1$. The left-hand side of this inequality is hard to deal with, because it is a random quantity. Therefore, we replace it by its mean value or its standard deviation. In both cases we find, for a sea-surface elevation that is Gaussian and that can be described by the Pierson–Moskowitz wave spectrum[4]

$$v\ll10c_0. \tag{17}$$

For realistic values of the windspeed, this condition is easily met. Details can be found in Appendix A.

B. The Complete Condition

For simplicity we take the Fraunhofer formula (Ref. 1, Eqs. 42 and 43) and find for a point close to the surface ($Z_R=\Delta z$):

$$p_b(t)=-\exp[ikD_0-i2k\cos\theta_s\zeta_s(t)]/D_0, \tag{18}$$

with $D_0=[X_R^2+(Z_T+\Delta z)^2]^{\frac{1}{2}}$. The direct wave for that same point is time-independent and given by

$$p_0=\exp(ikd)/d, \tag{19}$$

where $d=[X_R^2+(Z_T-\Delta z)^2]^{\frac{1}{2}}$. If now $\Delta z\ll Z_T$, we can put $D_0\approx d$ and

$$p_b(t)=-p_0\exp[-i2k\cos\theta_s\zeta_s(t)] \tag{20}$$

follows. This result into Eq. 12 yields

$$\sqrt{2}\cos\theta_s|\partial\zeta_s/\partial t|\ll c_0[1-\cos(2k\cos\theta_s\zeta_s)]^{\frac{1}{2}}. \tag{21}$$

Also this inequality has to be analyzed statistically. Using the mean-square criterium, plus some results of Appendix A, we obtain the condition

$$0.004v^2\ll c_0^2[1-\exp(-2k^2h^2\cos^2\theta_s)]. \tag{22}$$

Very low values of the windspeed (and consequently of h) need not be considered because the surface looses then its time dependency, and our problem disappears. Hence we assume that the quantity between square brackets in Eq. 22 has at least the value 10^{-5} (following from $k=1$, $v=1$ m/sec and $\cos\theta_s=0.6$). Then we find

$$v\ll0.05c_0\approx75\text{ m/sec}. \tag{23}$$

This condition is far more restrictive than the one found in the previous section, but still easy to satisfy.

III. SUMMARY

We have shown that for a medium with a time-dependent boundary the well-known Helmholtz equation holds only approximately. We have also derived a sufficient condition to which its use is subject: The time derivative of the surface elevation has to be much smaller than the speed at which the waves are propagating through the medium.

This condition has been analyzed statistically for the random sea surface. Using the Pierson–Moskowitz spectrum to describe the surface elevation, it is found that the wind speed has to be much smaller than the sound speed. This condition is very weak and easy to satisfy. Hence we conclude that it is permitted to use the Helmholtz equation in studies on sound scattering from the time-variant ocean surface.

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APPENDIX A: MEAN VALUE AND VARIANCE OF $|\partial\zeta/\partial t|$

It is assumed that the sea-surface elevation ζ is a stationary process with Gaussian probability density. It can then be shown (Ref. 5, p. 147) that also the random process $\partial\zeta/\partial t$ is stationary and Gaussian. Its mean value equals zero, and its variance is given by

$$\sigma^2 = \langle(\partial\zeta/\partial t)^2\rangle = -h^2\left[\frac{\partial^2}{\partial\tau^2}\Phi(0,0,\tau)\right]_{\tau\to 0}. \quad \text{(A1)}$$

The correlation function Φ can be expressed in terms of the surface-wave spectrum $A^2(\omega_s)$ (Ref. 6, p. 15):

$$\Phi(0,0,\tau) = (2h^2)^{-1}\int_0^\infty d\omega_s A^2(\omega_s)\cos(\omega_s\tau). \quad \text{(A2)}$$

Hence,

$$\sigma^2 = \frac{1}{2}\int_0^\infty d\omega_s A^2(\omega_s)\omega_s^2. \quad \text{(A3)}$$

For A^2 we take the Pierson–Moskowitz spectrum (Ref. 4, p. 1679). The integral can then be solved analytically:

$$\sigma^2 = 0.002v^2. \quad \text{(A4)}$$

Next we turn to the random process $Q \equiv |\partial\zeta/\partial t|$. Its probability density follows from Ref. 7, pp. 130, 131:

$$f(Q) = 2[\sigma(2\pi)^{\frac{1}{2}}]^{-1}\exp(-Q^2/2\sigma^2)U(Q), \quad \text{(A5)}$$

where U is the unit step function. Expected value and variance are readily obtained by integration:

$$E\{Q\} = \sigma(2/\pi)^{\frac{1}{2}}, \quad E\{Q^2\} = \sigma^2. \quad \text{(A6)}$$

Statistically, the condition $2Q \ll c_0$ means that we require

$$2\sigma \ll c_0. \quad \text{(A7)}$$

With Eq. A4 this yields

$$v \ll 5\sqrt{5}c_0 \approx 10c_0. \quad \text{(A8)}$$

[1] L. Fortuin, "The Sea Surface as a Random Filter for Underwater Sound Waves," J. Acoust. Soc. Am. **52**, 302 (1972).

[2] B. B. Baker and E. T. Copson, *The Mathematical Theory of Huygen's Principle* (Clarendon, Oxford, England, 1953), p. 8.

[3] The same result follows if we substitute the Fraunhofer formula (Ref. 1, Eqs. 42 and 43) into Eq. 13.

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