INVERSE WAVE PROPAGATION IN AN INHOMOGENEOUS WAVEGUIDE

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ABSTRACT

A solution is given for the problem of inverse propagation in an inhomogeneous rectangular two-dimensional waveguide. The sound speed is assumed to vary in depth and inverse propagation means the calculation of the field at range x_1 in terms of the field at range x_2 where $x_2 > x_1$. The method is analogous to that used by Wolf, Shewell, and Lalor for the inverse diffraction problem in a homogeneous half space. It is found that the field at x_1 can be expressed in terms of two integrals over the field at x_2. The kernel of the first integral is bounded and expresses physically the result at x_1 of the waves at x_2 reversing their direction of propagation and decay, ie they now propagate and decay in the direction of x_1. A reciprocity relation for this term is possible. The kernel of the second integral is singular and expresses the mathematical fact of the residual effect of the evanescent waves at x_1 as they reverse their direction at x_2 and now grow exponentially. Consequences of the neglect of this singular term are discussed.

INTRODUCTION

Recently, Wolf and Shewell[1] and Lalor[2] discussed the solution of the inverse diffraction problem in a homogeneous half-space. Simply, one has a field propagating into a half-space $z > 0$, and assumes the field is known on some plane $z = z_2$. The problem is then to find the field on the plane $z = z_1$ where $z_1 < z_2$. For example, one might wish to calculate the "near" field from the "far" field. The result is expressed as the inverse of one of the Rayleigh diffraction formulas. The kernel of the inversion contains two terms, one of which is singular. Methods for handling the singular term are discussed.

In this paper we briefly present a similar analysis with the problem being the calculation of the inverse field in a two-dimensional rectangular waveguide. Here, in addition, the waveguide is assumed to be inhomogeneous in the sense that the sound speed is a function of depth.

In Sec. 1 we present the basic analysis and express the field at $x_1 < x_2$ as a sum of two terms each of which is an integral over the field at x_2. The kernel of the first integral is bounded and the term describes that part of the field at x_1 due to waves at x_2 reversing their direction of propagation and decay. The kernel of the second integral is singular and the term describes exponentially growing waves at x_1 due to evanescent waves at x_2 which grow towards x_1. In Sec. 2 the reciprocity relation of the first term is derived, and in Sec. 3 a brief discussion is given of the consequences of neglect of the singular term.

1. GENERAL FORMALISM

In two dimensions the propagation of sound is governed by the Helmholtz equation

$$\phi_{xx} + \phi_{zz} + k^2\eta^2(z)\phi(x,z) = 0 \qquad (1)$$

for the velocity potential field ϕ.[3] Here, $\eta(z)$, the index of refraction, is proportional to the inverse of $c(z)$, the sound speed, and $k = 2\pi/\lambda$ is the wavenumber with λ the wavelength. Since c is a function of depth the equation is said to be inhomogeneous. The general problem of sound propagation involves the solution of (1) assuming that ϕ satisfies appropriate boundary conditions. Here we first wish to solve (1) in the region $0 \leq z \leq D$ and $0 \leq x < \infty$ (see Fig. 1), where ϕ satisfies boundary conditions at $z = 0$ and D, $x = 0$, and an outgoing radiation condition as $x \to \infty$. Then we will assume that the field is known on a (far) plane $x = x_2$ and express the field on a (near) plane $x = x_1 < x_2$ in terms of the field on x_2.

The solution of (1) is separable and can be written in terms of an infinite discrete spectral representation

$$\phi(x,z) = \sum_{j=0}^{\infty}{}' A_j\psi_j(z)\exp(ikm_jx) \qquad (2)$$

where the eigenfunctions ψ_j satisfy the ordinary differential equation

$$\psi_j'' + k^2[\mu_j - q(z)]\,\psi_j = 0 \qquad (3)$$

with

$$q(z) = 1 - \eta^2(z) \qquad (4)$$

and

$$m_j = \begin{cases} (1 - \mu_j)^{\frac{1}{2}} & 0 < \mu_j \leq 1 \\ +\, i(\mu_j - 1)^{\frac{1}{2}} & \mu_j > 1 \end{cases} \qquad (5)$$

The boundary conditions at $z = 0$ and D (which we do not specify) yield specific forms for the ψ_j and the discrete eigenvalues μ_j, which we assume for simplicity are confined to the positive real axis in the j-plane. The choice of branch in (5) is to ensure outgoing or decaying waves as $x \to \infty$. In addition we assume the ψ_j are orthonormal.

$$\int_0^D \psi_j(z)\psi_m(z)dz = \delta_{jm} = \begin{cases} 1 & j = m \\ 0 & j \neq m \end{cases} \tag{6}$$

Multiplying (2) by $\psi_i(z)$, integrating over z from 0 to D and using (6) yields

$$A_j = \exp(-ik\ m_j x) \int_0^D \phi(x,z)\psi_j(z)dz. \tag{7}$$

Now let $x = x_1$ and $z = z_1$ in (2), $x = x_2$ and $z = z_2$ in (7), and substitute the resulting (7) into (2) to get

$$\phi\ (x_1,z_1) = \sum_{j=0}^{\infty} \psi_j(z_1)\exp\left[\ ikm_j(x_1-x_2)\right] \int_0^D \psi_j(z_2)\phi(x_2,z_2)dz_2. \tag{8}$$

Next assume $x_1 < x_2$ and split the sum in (8) into two parts defined by

$$\sum{}^{-} = \sum_{j=0}^{J} \qquad \sum{}^{+} = \sum_{j=J+1}^{\infty} \tag{9}$$

where $\mu_J < 1$ and $\mu_{J+1} > 1$. To the result, add and subtract the term

$$\sum{}^{+} \psi_j(z_1)\exp\left\{ -k(\mu_j-1)^{\frac{1}{2}}(x_2-x_1) \right\} \int_0^D \psi_j(z_2)\phi\ (x_2,z_2)dz_2 \tag{10}$$

and rewrite the result as the sum of two terms

$$\phi(x_1,z_1) = \phi_1(x_1,z_1) + \phi_2(x_2,z_2) \tag{11}$$

where we define $(m = 1,2)$

$$\phi_m(x_1,z_1) = \int K_m(x_1,z_1;\ x_2,z_2)\phi\ (x_2,z_2)dz_2 \tag{12}$$

with

$$K_1(x_1,z_1;\ x_2,z_2) = \sum^{-} \psi_j(z_1)\psi_j(z_2)\exp[ikm_j(x_1-x_2)]$$

$$+\sum^{+} \psi_j(z_1)\psi_j(z_2)\exp[-k(\mu_j-1)^{\frac{1}{2}}(x_2-x_1)]$$

$$= \sum_{j=0}^{\infty} \psi_j(z_1)\psi_j(z_2)\exp[-ikm_j^*(x_2-x_1)] \tag{13}$$

where the * is complex conjugation, and

$$K_2(x_1,z_1;\ x_2,z_2) = \sum^{+} \psi_j(z_1)\psi_j(z_2)\exp[ikm_j(x_1-x_2)]$$

$$+\sum^{+} \psi_j(z_1)\psi_j(z_2)\exp[-k(\mu_j-1)^{\frac{1}{2}}(x_2-x_1)]$$

$$= \sum^{+} \psi_j(z_1)\psi_j(z_2)\sinh[k(\mu_j-1)^{\frac{1}{2}}(x_2-x_1)]\,. \tag{14}$$

Thus it is possible to write ϕ at (x_1,z_1) in terms of two integrals over ϕ at (x_2,z_2). The kernel of the first integral, K_1, is bounded and expresses physically the result at x_1 of the waves at x_2 reversing their direction of propagation and decay, ie they now propagate and decay in the direction of x_1. The kernel of the second integral, K_2, is singular since the summation in (14) goes to infinity, and the problem becomes ill-posed since a small change in the "initial" condition $\phi(x_2,z_2)$ could produce a large change in $\phi(x_1,z_1)$. This is the mathematical expression of the residual effect of the evanescent waves at x_2 as they reverse their direction and grow exponentially in the direction of x_1. The neglect of this latter term means neglect of large wavenumbers, short wavelength terms and hence an inability to gather information on an obstacle or process with a characteristic length smaller than a certain amount. There is thus a lower bound on the size of obstacles which can be seen.

2. RECIPROCITY

It is possible to express the ϕ_1 term as the inverse of one of the Rayleigh diffraction formulas. This is done as follows. The incoming wave Green's function $G^-(x,z;\ x',z')$ satisfies an equation similar to (1) with a delta function source term

$$G^-_{xx} + G^-_{zz} + k^2\eta^2(z)G^- = -\delta(x-x')\delta(z-z') \tag{15}$$

as well as the boundary conditions at $z = 0$ and D which are satisfied by the eigenfunctions, and the asymptotic condition of an incoming wave. It can be written as

$$G^-(x,z;\ x',z') = \sum_{j=0}^{\infty} \psi_j(z)\psi_j(z')G^-_j(x,x') \tag{16}$$

where G^-_j satisfies the differential equation

$$\left\{ \frac{d^2}{dx^2} + k^2(1-\mu_j)\right\} G^-_j(x,x') = -\delta(x-x') \tag{17}$$

and can be written as

$$G^-_j(x,x') = (2ikm^*_j)^{-1} \exp\{-ikm^*_j|x-x'|\} \tag{18}$$

where the complex conjugate of m_j is used in the exponential to ensure that for $j > J$ the function is decaying towards x_1. From (14) it can be easily seen that

$$K_1(x_1,z_1;\ x_2,z_2) = -2\frac{\partial}{\partial x_2} G^-(x_1,z_1;\ x_2,z_2) \tag{19}$$

so that ϕ_1 by (12) can be written as the inverse of a diffraction formula.

3. SUMMARY

To use these results one must be able to neglect the singular term ϕ_2. Neglect of ϕ_2 means neglect of terms of the order of $k(\mu_{J+1}-1)^{\frac{1}{2}}$ and larger, ie high frequency terms. The term $k = \omega/c$ where c is some reference sound speed, eg the sound speed at the surface. This establishes a characteristic length $L = \lambda/2\pi(\mu_{J+1}-1)^{\frac{1}{2}}$ below which we cannot measure. The higher the frequency of sound the smaller the obstacles we can see, but high frequency sound is rapidly attenuated in the ocean anyway, so that neglect of ϕ_2 probably yields no worse results than are now available.

Footnotes

* Temporary Address

1. E. Wolf and J. R. Shewell, Phys. Lett. 25A, 417 (1967) and 26A, 104 (1967)
2. E. Lalor, J. Math. Phys. 9, 2001 (1968) and J. Opt. Soc. Am. 58, 1235 (1968). These papers also consider similar mathematical questions which arise here in greater detail.
3. The harmonic time dependence exp(-iωt) is assumed throughout.

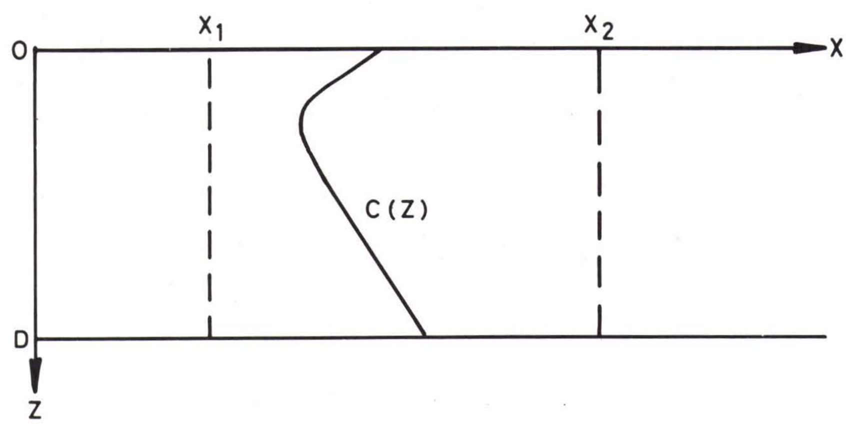

FIG. 1 INVERSE PROPAGATION IN A RECTANGULAR TWO-DIMENSIONAL WAVEGUIDE. THE SOUND SPEED C IS A FUNCTION OF DEPTH Z. THE FIELD IS ASSUMED KNOWN ON THE PLANE $X = X_2$ AND THE PROBLEM IS TO CALCULATE IT ON THE PLANE $X = X_1$