RAYS AND STATISTICAL DIFFRACTION THEORY

by

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ABSTRACT

A method of extending ray tracing is proposed, such that medium-scale irregularities of a statistical nature are taken into account along with large-scale irregularities in refractive index of the medium.

INTRODUCTION

Ray tracing is a relatively simple and very practical method of obtaining solutions to the fundamentally very difficult problem of wave propagation in an inhomogeneous medium. These solutions are admittedly approximate, but give perfectly satisfactory "engineering" answers, provided the scale size of the inhomogeneities is large compared to the propagating wavelength, and provided one does not require answers in regions of focusing or shadowing.

I want to address myself to the problem of dealing with irregularities, both in the medium and on the boundaries, whose scale size is not large compared to the propagating wavelength. In particular, I am interested in scale sizes of the same order or somewhat larger than the propagating wavelength. I shall exclude from consideration those irregularities whose scale size is smaller than the propagating wavelength, since in a sense these will be invisible to the propagating wave and are likely to have only a collective effect.

(For example, bubbles will have the effect of changing the acoustic properties of the water when encountered by metre wavelengths, and so the problem reverts to that of the effect of conglomerates of such bubbles of scale sizes of a wavelength or larger.)

One can state the problem in a somewhat more restricted fashion by asking how one can retain the advantages of using ray tracing to deal with large-scale irregularities when medium-scale irregularities are also present. I eventually want to discuss the effect of medium-scale irregularities on the sea surface, in the depth of the thermocline and in the medium everywhere. But it is useful to start with the rough surface.

EXAMPLE OF SURFACE ROUGHNESS

After a more-or-less tortuous path, some of the rays leaving the source will strike the surface [Fig. 1]. A straightforward extension of the ray tracing concept would be to say that each ray is reflected in the local specular direction. But there are two objections to this course: (1) this idea is only valid for large-scale irregularities and (2) a hideously large number of ray tracings would have to be made in order to obtain a satisfactory statistical ensemble.

An alternative approach, and the one I shall advocate, is to use ray tracing to just below the surface; then to employ statistical diffraction theory to account for the effect of the statistically rough surface; and then to employ ray tracing again to describe the subsequent progress of the field. This gets over the two disadvantages I mentioned in connection with the first approach: (1) since diffraction is taken into account the validity of the method is not restricted to large-scale irregularities, and (2) the ensemble averaging is performed <u>at</u> the surface and so a single ray tracing suffices to describe the subsequent reflected field.

I shall give more details of this in a moment. But first, I must describe more clearly how one can make the transition from rays to fields and back again.

RAYS AND PLANE WAVES

Dating from Rayleigh's treatment of the problem of reflection from a corrugated surface [Ref. 1], the expansion of acoustic fields in terms of plane waves travelling in different directions has become increasingly popular. Intuitively, one expects there to be an equivalence between such plane waves and the purely geometrical concept of rays; that equivalence will now be demonstrated.

Uniform Medium

The pressure field $p(x,y,z)$ in the half space $z \geq 0$ can be represented [Ref. 2] by the plane-wave spectrum $F(\alpha, \beta)$, such that

$$p(x,y,z) = \iint_{-\infty}^{\infty} F(\alpha, \beta) \exp\{-jk_0(\alpha x + \beta y + \gamma z)\} \, d\alpha \, d\beta \qquad \text{[Eq. 1]}$$

where (α, β, γ) are the cosines of the angles formed between the direction of a single plane-wave component and the three rectangular coordinate axes (x,y,z), and k_0 is the phase constant (wave-number) of the medium.

Assuming the acoustic source to be at the origin, at a large distance r from the source, such that $k_0 r >> 1$, it can be shown by applying stationary phase methods that the pressure field is asymptotically

$$p \quad j\frac{\lambda\gamma}{r} F(\alpha, \beta) \; e^{-jkr} \, . \qquad \text{[Eq. 2]}$$

Thus the angular plane-wave spectrum $F(\alpha, \beta)$ is proportional to the directivity pattern of the source.

The associated intensity is

$$I = \frac{|p|^2}{2Z_0} = \frac{\lambda^2 \gamma^2 |F(\alpha, \beta)|^2}{2Z_0 r^2}$$

where Z_0 is the characteristic impedance of the medium. It will be useful later to refer intensities to the "intensity at unit distance", which is

$$\mathfrak{J}_0 = \frac{\lambda^2 \gamma^2 |F(\alpha, \beta)|^2}{2Z_0} . \qquad \text{[Eq. 3]}$$

<u>Layered-Inhomogeneous Medium</u>

When the acoustic properties of the medium change with z, then the angular spectrum $F(\alpha, \beta)$, which describes the field at the source level $z = 0$, no longer describes the field for any other z. However, by writing the angular spectrum as a function of z, namely $F(\alpha, \beta, z)$, it can be supposed that all plane waves emanating from the source follow the ray paths prescribed by geometrical acoustics, provided the irregularities in the medium are of scale-size very large compared to the acoustic wavelength. Hence [Ref. 3],

$$p(x,y,z) = \iint_{-\infty}^{\infty} F(\alpha, \beta, z) \exp\left\{-jk_0\left(\alpha x + \beta y + \int_0^z q\, dz\right)\right\} d\alpha\, d\beta \qquad \text{[Eq. 4]}$$

where $q = n \cos \theta$, the refractive index $n = n(z) = c(0)/c(z)$ is the ratio of the sound speeds, and θ is the local angle to the vertical of the ray path [Fig. 2]. From Snell's law it is obvious that

$$q^2 = n^2 - \alpha^2 - \beta^2 . \qquad \text{[Eq. 5]}$$

The W.K.B. solution [Ref. 4] for a plane wave travelling in a layered-inhomogeneous medium yields

$$F(\alpha,\beta,z) = \left(\frac{\rho c \cos\theta_0}{\rho_0 c_0 \cos\theta}\right)^{1/2} F(\alpha,\beta) \qquad \text{[Eq. 6]}$$

where $\rho = \rho(z)$ is the density, and the subscript 0 refers to the source level. (Note that $Z = \rho c$ is the characteristic impedance at the level z).

Integrating Eq. 4 by stationary phase methods, asymptotically

$$p \; j \frac{\lambda}{\Delta^{1/2}} F(\alpha_0, \beta_0, z) \; \exp\left\{-jk_0(\alpha_0 x + \beta_0 y + \int_0^z q\, dz)\right\} \qquad \text{[Eq. 7]}$$

where (α_0, β_0) are the direction cosines at the source which satisfy the stationary-phase conditions. These conditions are that

$$x = -\int_0^z \frac{\partial q}{\partial \alpha}\, dz \qquad \text{[Eq. 8a]}$$

and that

$$y = -\int_0^z \frac{\partial q}{\partial \beta}\, dz \qquad \text{[Eq. 8b]}$$

which are just the equations of the ray path. The quantity Δ is a determinant in the general case [Ref. 3], but in the x-z plane reduces to

$$\Delta = \left(\int_0^z \frac{\partial^2 q}{\partial \alpha^2} dz\right)\left(\int_0^z \frac{\partial^2 q}{\partial \beta^2} dz\right) . \qquad \text{[Eq. 9]}$$

The intensity corresponding to the pressure of Eq. 7 is

$$I = \frac{|p|^2}{2Z} = \frac{\lambda^2 \, |F(\alpha,\beta,z)|^2}{2Z\,\Delta} \qquad \text{[Eq. 10]}$$

where, as can be shown by applying Eq. 9 to Eq. 5,

$$\Delta = \frac{x}{\sin^2 \theta_0} \int_0^z \frac{\sin \theta}{\cos^3 \theta} dz \quad . \qquad \text{[Eq. 11]}$$

Then, the final intensity formula is

$$I = \mathfrak{I}_0 \frac{\sin^2 \theta_0}{x \cos \theta \cos \theta_0 \int_0^z \frac{\sin \theta}{\cos^3 \theta} dz} \qquad \text{[Eq. 12]}$$

and is equivalent to the formula developed by Krol [see Session 2 of these Proceedings] using purely geometrical arguments.

Thus an equivalence between a ray description and a plane-wave description of acoustic fields in a layered inhomogeneous medium has been established, which takes care of the effect of large-scale irregularities.

The next step in the argument is to consider how each of these plane-wave components, which go to make up the total field, are affected by medium-scale statistical irregularities encountered either in the medium or at its boundaries.

STATISTICAL DIFFRACTION THEORY

Consider the simplest case, shown in Fig. 3, of a plane wave incident normally on a "random phase screen". Such a screen alters the phase of a wave propagating through it in a random manner, but leaves its amplitude unchanged. (The physical mechanisms in the ocean which produce such random phase screens will be discussed in the next section.)

If the random phase induced by the screen is a zero-mean, gaussian random process of variance σ_Φ^2, then the transmitted field will consist of a coherent part and an incoherent part. (Coherence is used here in the sense of the phase having a deterministic relation to the incident phase.) It can be shown [Ref. 5] that the coherent part of the transmitted field is a plane wave, in all respects the same as the incident field, except that its amplitude is reduced by $\exp\{-\frac{1}{2}\sigma_{\Phi}^{2}\}$. This can be expressed by saying there is a "coherence loss" of intensity of

$$\exp\{-\sigma_{\Phi}^{2}\} \quad \text{or} \quad 4.34\,\sigma_{\Phi}^{2}\ \text{dB} .$$

But this is not a real, absorptive loss, and the remaining transmitted energy is incoherently scattered in a pattern which is determined by the second-order (i.e., lateral correlation) statistics of the phase across the screen.

In terms of rays: the incident ray suffers a "loss", but apart from that continues as though the screen were not there. The lost energy is converted at the screen into new, incoherent sources of energy whose angular pattern can be determined. Ray tracing can be applied to follow the subsequent behaviour of this new source of acoustic energy.

RANDOM MECHANISMS IN THE SEA

Rough Sea Surface

For a plane wave incident obliquely on a randomly rough sea surface, [Fig. 4], the simplest (and most common) approach is to ignore amplitude effects and to consider only the random phase induced in the incident wave arising from the local excess path travelled by the wave to and from the surface, compared with reflection from the mean surface. Thus the surface is replaced by a random phase screen. If the surface profile is a zero-mean, gaussian random process of variance σ_h^2 the random phase variance is

$$\sigma_{\Phi}^{2} = (2k\cos\theta)^{2}\,\sigma_h^{2} .$$

Thus the incident ray is specularly reflected with a coherence loss of $4.34\,\sigma_{\Phi}^{2}$, and the remaining energy is scattered incoherently with an intensity pattern determined by the spatial correlation function of the surface roughness.

Internal Waves

Figure 5 shows an idealized model of an abrupt thermocline boundary separating two regions of the ocean in which the sound velocities, and hence the phase constants, k_1 and k_2, are slightly different. If the boundary profile is a zero-mean, gaussian random process of variance σ_h^2, then the same sort of arguments used for the rough surface establish the first-order effect on an obliquely incident plane wave of such a boundary as a random phase screen with phase variance

$$\sigma_{\Phi}^{2} = (k_2 - k_1)^2 \sec^2\theta\, \sigma_h^2 .$$

(A similar expression has been used to examine the effect of irregularities in dielectric holograms [Ref. 6].)

Volume Irregularities

If a plane wave travels a distance ℓ [see Fig. 6] through a slab of tenuous irregularities in refractive index, then it is physically plausible to suppose that the emerging field is randomly modulated in phase but unaltered in amplitude. (For a more rigorous validation of this approach, see the résumé of the work of Fejer and Bramley in Ref. 5.) Hence the slab of irregularities behaves as a random phase screen. If ℓ is many times a typical scale size, ς_0, of the irregularities, then a crude application of the Central Limit Theorem establishes that the emerging phase is approximately gaussian, with variance

$$\sigma_{\Phi}^{2} = k^2 \varsigma_0 \ell\, \sigma_n^2$$

where σ_n^2 is the variance of the refractive index fluctuations. Hence as a ray traverses such irregularities it will suffer a loss of

$$4.34\,k^2\,\zeta_0\sigma_n^2 \quad \text{dB/unit length}$$

of its energy to incoherent scatter, the angular spread of which will be determined — as in the other examples — by the lateral scale size of the irregularities.

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DISCUSSION

The author confirmed that these ideas could be applied to a surface sinusoid with roughness superimposed, and also to a rough and randomly layered bottom - although the latter is more difficult.

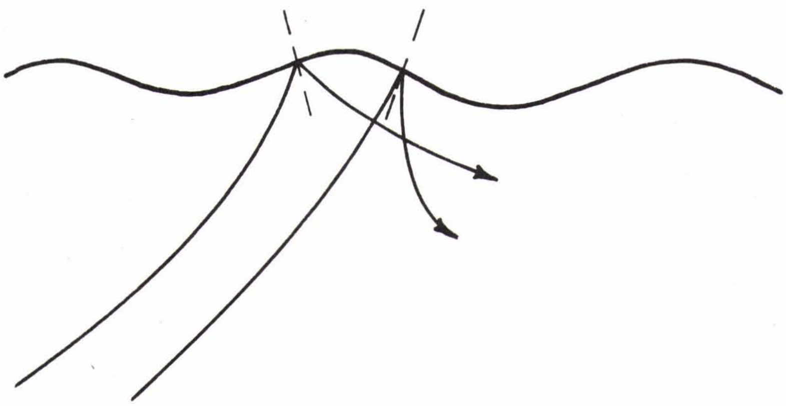


FIG. 1

FIG. 2

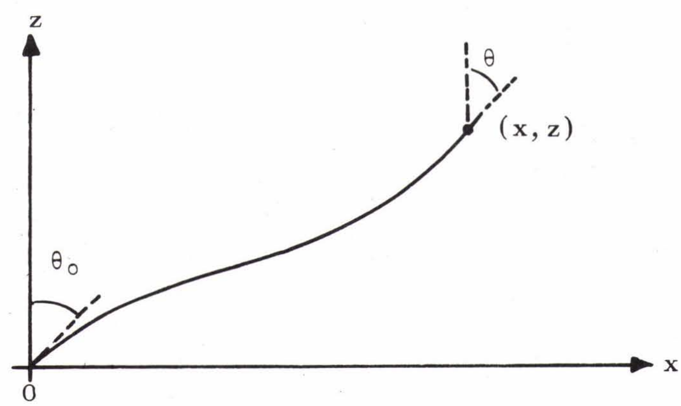
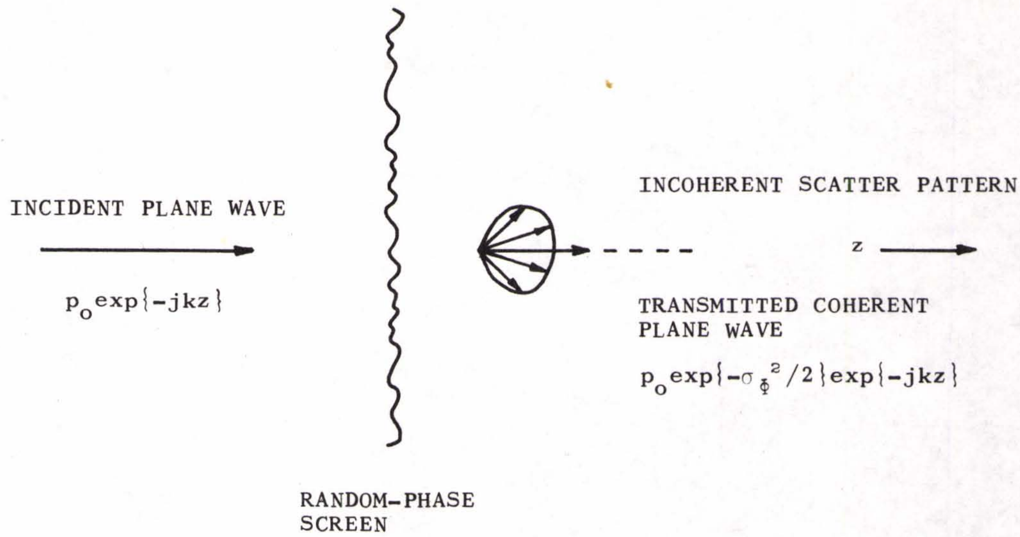


FIG. 3



ROUGH SURFACE

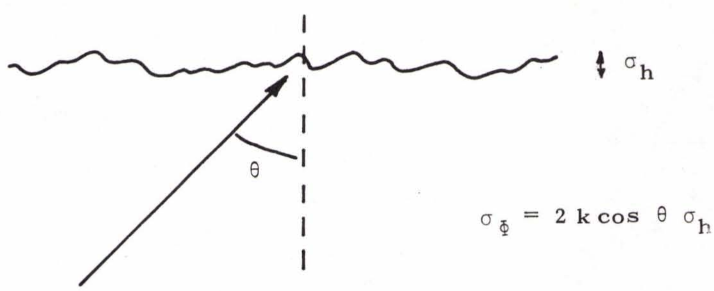
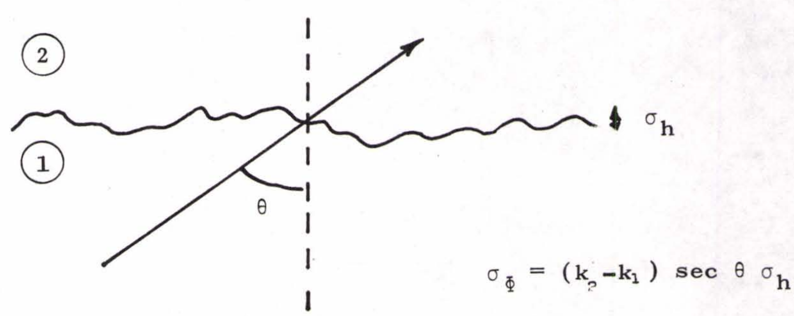


FIG. 4

INTERNAL WAVES

FIG. 5



VOLUME IRREGULARITIES (MEDIUM SCALE)

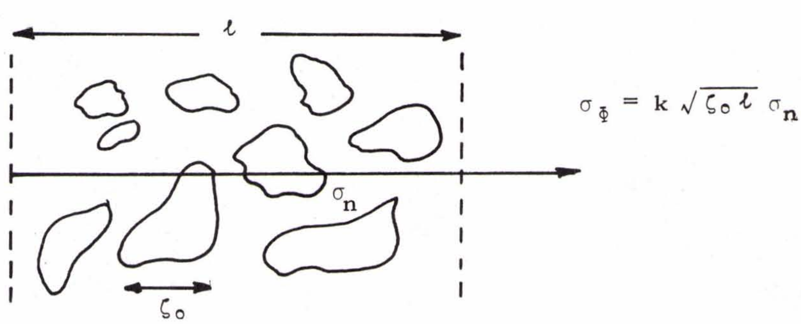


FIG. 6