RANGE DEPENDENT NORMAL MODES IN UNDERWATER SOUND PROPAGATION

by

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ABSTRACT

Normal mode theory is best suited for the case of stratified media. Range dependence of the medium properties and of its boundary may nevertheless be taken into account in the framework of an adiabatic approximation provided the changes with range are sufficiently gradual. We have extended this approach and have included possible range variations of the boundaries. To lowest order, the solution furnished by this method consists of the depth functions of a locally stratified medium; in higher order, the range functions satisfy a system of coupled equations, with the coupling terms causing an exchange of energy between modes. As an application, we have evaluated acoustic fields in an isovelocity wedge-shaped ocean (continental shelf) using the normal-mode method with adiabatic range variation, obtaining good agreement with the exact solution due to Bradley and Hudimac.

Normal mode theory of underwater sound propagation, as applied in the usual way, is useful for the case of stratified media where the wave equation separates. A range dependence of the medium properties (sound velocity profile) and of its boundaries may nevertheless be taken into account in the framework of an adiabatic approximation, provided the changes with range are sufficiently gradual. We have extended this approach, which was first indicated by Pierce[1] and by Milder[2] for the variable medium, and have included possible range variations of the boundaries. To lowest order, the solution furnished by this method contains the depth functions of a locally stratified medium, whose eigenvalues k_n at each range point enter in an equation for the range function that replaces the range function $H_o^{(1)}(k_n\rho)$ of the stratified case. To higher order (approaching the exact case), the range functions satisfy a system of coupled equations, with the coupling terms causing an exchange of energy between modes.

Starting from the wave equation corresponding to a point source in an inhomogeneous medium,

$$\nabla^2\phi(\vec{r}) + k^2(\vec{r})\,\phi(\vec{r}) = \delta(\vec{r}-\vec{r}_o) \tag{1}$$

where $k(\vec{r}) = \omega/c(\vec{r})$, one attempts a "separation" in horizontal and vertical coordinates $\vec{r} = (\vec{\rho}, z)$ of the form

$$\phi(\vec{r}) = \sum_n \psi_n(\vec{\rho})\,u_n(z,\vec{\rho})\,; \tag{2}$$

the "local depth functions" $u(z,\vec{\rho})$ satisfy the usual depth equation

$$\partial^2 u_n(z,\vec{\rho})/\partial z^2 + [k^2(z,\vec{\rho}) - k_n^2(\vec{\rho})]u_n(z,\vec{\rho}) = 0, \tag{3}$$

whose modal eigenvalues $k_n^2(\vec{\rho})$, determined from the boundary conditions at the boundaries $z = z_{\pm}(\vec{\rho})$, are now range dependent. Inserting (2) in (1) leads to a set of coupled range equations (source at $\vec{\rho} = 0$, $z = z_o$):

$$\begin{aligned}[\nabla_\rho^2 + k_n^2(\vec{\rho})]\psi_n(\vec{\rho}) &= \delta(\vec{\rho})u_n(z_o,0) \\ &\quad - 2\sum_m [\vec{\nabla}_\rho \psi_m(\vec{\rho})]\cdot\int_{z_-}^{z_+} u_n(z,\vec{\rho})\vec{\nabla}_\rho u_m(z,\vec{\rho})dz \\ &\quad - \sum_m \psi_m(\vec{\rho})\int_{z_-}^{z_+} u_n(z,\vec{\rho})\nabla_\rho^2 u_m(z,\vec{\rho})dz.\end{aligned} \tag{4}$$

This is still an <u>exact</u> system of equations, but the lack of separability has led to the appearance of mode coupling terms, which will however be small for sufficiently gradual range dependence.

For the important special case where range variations take place in one horizontal direction only (x, say), the range function may be written as a Fourier integral,

$$\psi_n(\vec{\rho}) = \int_{-\infty}^{\infty} g_n(x,k_y)e^{ik_y y}dk_y/2\pi \tag{5}$$

Inserting in (4) and neglecting mode coupling, one finds the range equation

$$\{d^2/dx^2 + [k_n^2(x) - k_y^2]\}g_n(x,k_y) = u_n(z_o,0)\delta(x). \tag{6}$$

We first solved the special case of an isovelocity wedge-shaped ocean, with the origin at the shore, the source at $\vec{r}_o = (x_o, 0, z_o)$, and the ocean floor given by $z_+ \equiv h(x) = h_o(x/x_o)$, h_o being the ocean depth at the source location. We obtained the exact solution (without mode coupling)

$$\phi(\vec{r}) = (x_o/2ih_o)\sum_{n=0}^{\infty} \sin[(n+\tfrac{1}{2})\pi z/h]\,\sin[(n+\tfrac{1}{2})\pi z_o/h_o] \cdot \int_{-\infty}^{\infty} J_{\nu_n}(k_x x_<)\, H^{(1)}_{\nu_n}(k_x x_>)\,\exp(ik_y y)\,dk_y \tag{7}$$

where $x_{\gtrless}$ = max, min (x, x_o), and

$$\nu_n = \left\{ \tfrac{1}{4} + [(n+\tfrac{1}{2})\pi x_o/h_o]^2 \right\}^{1/2} \tag{8}$$

If $x_> \gg x_<$, a saddle-point evaluation of (7) gives

$$\phi(\vec{r}) = (x_o/h_o r)\exp(ikr)\sum_{n=0}^{\infty} \sin[(n+\tfrac{1}{2})\pi z/h]\,\sin[(n+\tfrac{1}{2})\pi z_o/h_o] \exp(-\tfrac{1}{2}i\pi\nu_n)\,J_{\nu_n}(kx_> x_</r), \tag{9}$$

where $r = (x_>^2 + y^2)^{\frac{1}{2}}$.

As a numerical example, we chose a free-surface, rigid-bottom wedge with $h_o/x_o = 0.2$, with the source located at $x_o = 25\lambda$, $z_o = (1/3)h_o$. The acoustic intensity $|\phi|^2$ obtained from (9) is shown in Fig. 1 in the plane y = 0, plotted vs. x/λ between 0 and 10. Modes cut off at positions indicated by arrows. Contour lines represent intensity variations in steps of 3 dB for three contours of highest intensity, and 6 dB otherwise. The results are compared in Fig. 2 with $|\phi|^2$ calculated from the exact solution of the same wedge problem, as found by Bradley and Hudimac[3], and likewise evaluated by the saddle point method. Small differences can be observed between the lower right portion of the figures, and are probably attributable to mode coupling effects.

A computer program has been developed by us for solving the adiabatic range equation (Eq. (4) without coupling) for realistic sound velocity profiles with arbitrary (but gradual) variations in range, and similar variations of the ocean floor[4]; this is now being extended to the system of Eqs. (4) including the coupling. The same method is used for the depth equation (3) also, in order to apply a unified treatment to all parts of the problem.

To solve the depth equation, we divide the ocean into P horizontal layers (P $\sim$ 10 for practical purposes), and linearize the wave number $k^2(z,\vec{\rho})$ at each range point $\vec{\rho}$, so that in the p^{th} layer

$$k_p^2(z,\vec{\rho}) = \alpha_p(\vec{\rho}) + \beta_p(\vec{\rho})\left[z_{p-1}(\vec{\rho}) - z\right], \qquad (10)$$

z_{p-1} being the interface between layers p-1 and p. With the new variable in the p^{th} layer

$$\zeta_{np}(z) = \left[\beta_p^2(\vec{\rho})\right]^{-1/3}\left\{-\alpha_p(\vec{\rho}) + \beta_p(\vec{\rho})\left[z - z_{p-1}(\vec{\rho})\right] + k_n^2(\vec{\rho})\right\}, \qquad (11)$$

the solutions of (3) are the Airy functions[5]

$$u_{np}(z,\vec{\rho}) = A_{np}(\vec{\rho})\,Ai\left(\zeta_{np}(z)\right) + B_{np}(\vec{\rho})\,Bi\left(\zeta_{np}(z)\right). \qquad (12)$$

The boundary conditions at each interface, i.e. $u_{np}(z,\vec{\rho})$ and $\partial u_{np}(z,\vec{\rho})/\partial z$ to be continuous, permit to evaluate all the coefficients A_{np}, B_{np} as follows. At each interface, A_{np} and

B_{np} may be expressed by A_{np-1} and B_{np-1}. The one condition at the ocean surface z=0, namely $u_{n1}(0,\vec{\rho}) = 0$, determines A_1, while B_1 may be fixed by the overall normalization of u_n. At the ocean floor, a decaying exponential for u_{nP+1} ($B_{nP+1} \equiv 0$) matched to u_{nP} determines A_{nP+1}, while matching of the derivative furnishes the eigenvalue equation for $k_n(\vec{\rho})$.

The analogous treatment for the uncoupled range equation, in the form of e.g. Eq. (6), now linearizes the quantity $k_n^2(x) - k_y^2$, where $k_n(x)$ is obtained by solving the eigenvalue equation at the boundaries x_m of the range intervals, the subdivisions ranging from x_{-M} to x_M (with the source at $x = x_0 \equiv 0$):

$$k_n^2(x) - k_y^2 = a_m + b_m\,(x_{m-1} - x). \tag{13}$$

With the new variable

$$r_{nm}(x) = \left[b_m^2\right]^{-1/3}\left[b_m\,(x - x_{m-1}) - a_m\right], \tag{14}$$

one again has the Airy function solutions

$$g_{nm}(x) = A_{nm}\,Ai\,(r_{nm}) + B_{nm}\,Bi\,(r_{nm}). \tag{15}$$

Boundary conditions in the adiabatic case are again the continuity of $g_{nm}(x)$ and $dg_{nm}(x)/dx$ at each interface. At $x > |x_{\pm M}|$, however, one now has a radiation condition which requires outgoing waves only as $x \to \pm\infty$, while in all finite intervals in the region $x < |x_{\pm M}|$, the solutions (15) represent both in- and outgoing waves, so that in contrast to the now-fashionable

PE (parabolic equation) method[6], the possibility of backscattering towards the source is always present. Modes that have cut off by the time they reach $x = x_{\pm M}$, are matched to decaying exponentials, of course. The coefficients of the outgoing or decaying exponentials in $x > x_M$ or $x < x_{-M}$ are denoted by A_{nM+1} or $A_{n,-M-1}$, respectively .

If the coefficients of the solution (15) are normalized by

$$\begin{aligned} \alpha_{nm} &= A_{nm}/A_{n,M+1}, \quad \beta_{nm} = B_{nm}/A_{n,M+1} \qquad (m>0), \\ \alpha_{nm} &= A_{nm}/A_{n,-M-1}, \quad \beta_{nm} = B_{nm}/A_{n,-M-1} \qquad (m<0), \end{aligned} \tag{16}$$

then the mentioned radiation condition determines $\alpha_{n,\pm M}$ and $\beta_{n,\pm M}$ completely, and by further matching at all successive boundaries of segments, all other coefficients are determined down to $\alpha_{n,\pm 1}$ and $\beta_{n,\pm 1}$. Accordingly, the only two unknowns left are the common denominators $A_{n,M+1}$ and $A_{n,-M-1}$. These cannot be determined from the normalization, but from the requirements that the solution of Eq. (6), which actually is the Green's function of the problem, (i) be continuous at $x = 0$, and (ii) have a discontinuity in slope such that the source strength in (6) is reproduced. Satisfying these condition completely solves the range-dependent problem (in the case of x-dependence only).

This approach is now being programmed by us, together with the case of cylindrical-coordinate range variation (ρ dependence only). The latter solution will be utilized for "patching up" the solution of the PE method in regions where the latter becomes unreliable, either due to equivalent ray angles exceeding inclinations of $\sim 20^{\circ}$ (i.e. propagation up-slope, or over a seamount), or near the source. The ρ-dependent case is, however, less general than the x-dependent one, since it cannot describe e.g. sound propagating up the continental shelf at an angle, and being deflected back out to sea. Numerical results of our range-dependent normal-mode program will be published elsewhere.

References

*Supported by the Office of Naval Research, Code 486.

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4. For a WKB approach to this problem, cf. F. Ingenito, Proceed. Saclant Cen. Conference on Sound Propagation in Shallow Water, Sept. 1974.

5. Cf., e.g., D.C. Stickler, J. Acoust. Soc. Am. 57, 856 (1975).

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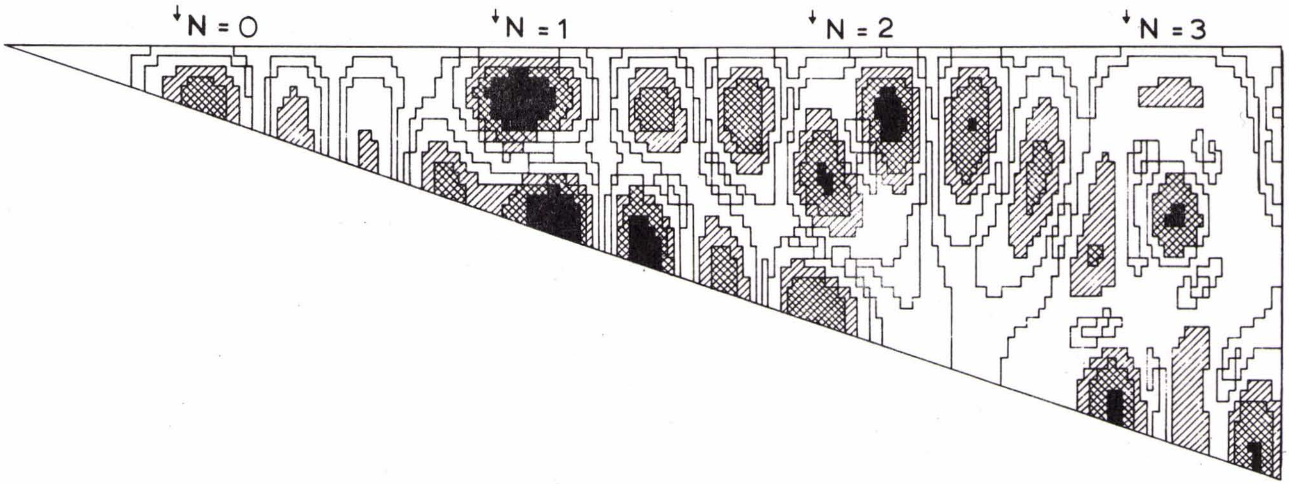

FIG. 1

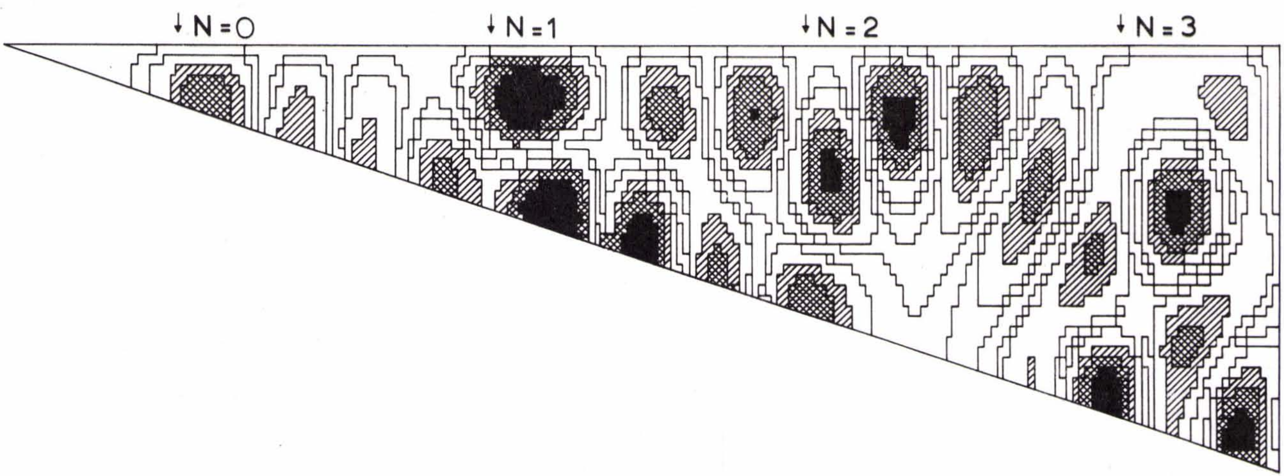

FIG. 2