HAMILTONIAN METHODS IN HYDRO-ACOUSTIC PROPAGATION

by

B.O. Koopman
Arthur D. Little, Inc.
Cambridge, Mass., U.S.

The investigation that I wish to outline here originated in the military need of forecasting detection probabilities of underwater acoustic emitters, in cases in which only a somewhat general idea of the sound speed profile is available, based, for example, on a known geographical location and average sound speed behaviour at a given time of year. Since detection of a distant object is the objective, only very faint signals enter — certainly nothing that could produce the non-linear effects, shock-waves, etc., that have been discussed in many of the papers reported here. Evidently the object of present interest is not this or that result of meticulously accurate computations based on exact knowledge of the sound speed c as a function of position, but more general facts that are relatively <u>stable</u> — i.e., are not radically altered by slight changes in the function c. Moreover, it is not only necessary that the stable evaluations be rather rough approximations (since we cannot know values of c in an area of future enemy operations except roughly) — but this is <u>sufficient</u> for military applications.

These requirements lead us in two directions: quantitative <u>generalities</u>: mathematically this means <u>theorems</u> rather than detailed computations; and a <u>statistical</u> attribute of the results. Our situation suggests a similar one in the quantitative study of those other complex physical systems, composed of the molecules of

---

* The investigation reported here grew out of an earlier phase, supported by the U.S. Naval Ordnance System Command. The author is uniquely responsible for all views expressed herein.

a gas; and just as statistical mechanics bases its methods on Hamiltonian theory, phase-space and its integral invariants, so we shall find, rather surprisingly, that similar branches of classical mathematics will play an essential part in our investigations.

It is not too surprising that the methodology of classical Hamiltonian dynamics should enter our problems of hydro-acoustic propagation: mathematics does not know the difference between Fermat's principle of least time and Maupertuis' principle of least action, which led immediately to the Hamiltonian theory. But let us start from the beginning.

Our starting point is D'Alembert's wave equation in the velocity potential ψ

$$\nabla^2\psi - \psi_{tt}/c^2 = 0$$

(with possible slight modification in derivatives of lower order) together with the energy density expression

$$E = \frac{\rho_0}{2}\,[\,|\nabla\psi|^2 + |\psi_t|^2/c^2]$$

and the energy flux vector

$$\vec{F} = -\rho\,|\bar{\psi}_t\,\nabla\psi|\ .$$

Note that we are using the absolute values, to allow for complex wave functions ψ. We recall that for every c which is a function of geometrical position and is <u>independent of time t,</u> the wave equation has a consequence that an equation of continuity is satisfied; i.e., that

$$\partial E/\partial t + \nabla\cdot\vec{F} = 0\ .$$

Geometrical acoustics is a valid approximation at sufficiently high frequencies: $\omega \gg c/\text{depth}$, where $\omega = 2\pi \times$ frequency. The element which transports hydro-acoustic energy is the <u>travelling</u>

wave, that is, a solution of the wave equation of the form

$$\psi = uf[\omega(t - S)], \qquad S = S(x,y,z) \ .$$

Here the function $f = f(x)$ must be defined for all x. For monochromatic steady state propagation, we take $f(x) = e^{ix}$, while for an infinite pulse, $f(x) = \delta(x)$, the Dirac delta function, etc. The coefficient u depends on x, y, z, t, and even ω; but it is thought of as varying "slowly" with these quantities. When the above ψ is inserted in the wave equation and only terms in ω^2 retained, the eikonal equation is obtained,

$$|\nabla S|^2 - 1/c^2 = 0 \ .$$

A solution $S = S(x,y,z)$ of this, when set equal to a constant has for locus a wave front; and the family of such loci: $S(x,y,z) = t$ is a moving surface as the time t increases — the wave front of the travelling wave. In space-time it is a characteristic hypersurface of the wave equation.

The eikonal equation has, in its turn, characteristic manifolds, the bicharacteristics of the wave equation or rays. The classical theory of all these relations — known for well over a century — gives us the rule for writing the differential equations of the latter. We replace $\partial S/\partial x$, etc., in the eikonal equation by p_x, etc., and set

$$H = \tfrac{1}{2}\left(p_x^2 + p_y^2 + p_z^2 - \frac{1}{c^2}\right) \ ,$$

so that the eikonal equation could be written as $H = 0$. Then we write the system of six differential equations

$$d\tau = \frac{dx}{H_{p_x}} = \frac{dy}{H_{p_y}} = \frac{dz}{H_{p_z}} = \frac{dp_x}{-H_x} = \frac{dp_y}{-H_y} = \frac{dp_z}{-H_z}$$

where the denominators are the partial derivatives of H with respect to the six independent variables (x, y, z, p_x, p_y, p_z), and τ is a parameter.

Clearly the equations for the bicharacteristics are in Hamilton's canonical form:

$$\frac{dx}{d\tau} = \frac{\partial H}{\partial p_x} = p_x, \text{ etc.}, \quad \frac{dp_x}{d\tau} = -\frac{\partial H}{\partial x} = \frac{\partial}{\partial x}\frac{1}{2c^2}, \text{ etc.}$$

They are the equations of the motion of a particle of unit mass, solicited by a force derived from the potential $-1/2c^2$, referred to a "time" parameter τ. We shall call it the pseudo-particle and τ the pseudo-time.

If in these equations the three "momentum" variables p_x, p_y, p_z are eliminated, we find a system of three differential equations of the second order. If, finally, we replace the independent variable $d\tau$ by the physical time t by means of the relation $d\tau = c^2 dt$, our equations become identical with the differential equations of the rays, as obtained by minimizing the time $\int ds/c$ in Fermat's principle.

Further relationships now become clear: We have $p_x = dx/d\tau = dx/cds = \cos\alpha/c$, $\cos\alpha$ being the first direction cosine, etc. Thus the "momenta" are directional quantities along the rays. The pseudo-velocity is seen to be $ds/d\tau = 1/c = \nu$, the refractive index. This is all part of the wave-particle complementarity, which worried physicists as far back as the Newton-Huygens arguments about light.

What is the relationship between the rays and the hydro-acoustic energy? The answer is given by going back to the expressions for the density of energy E and its flux vector density $\vec{F}$, and replacing ψ by its travelling wave expression. On discarding lower powers of ω and then using the eikonal equation, we find simple expressions for these quantities in terms of the intensity $|u|^2$; the fact that the energy flux vector $\vec{F}$ is in the direction of the momentum (p_x, p_y, p_z), that is, of the tangent to the rays, becomes evident from this substitution. Finally, the equation of continuity obeyed by $E, \vec{F}$ leads to the following one in the present picture: If the pseudo-time τ is used instead of t, and if p_x, p_y, p_z are regarded as the components of a fluid velocity field (based on the function S we are using), then the energy (in these units) obeys the classical equation of continuity.

We may forecast one of our results in the following terms: If in the above picture of a spatial flow, the fluid were incompressible, then the energy density would be a "first integral" of the ray equations; i.e., it would remain constant along each ray, so that, by tracing it to its source (the emitter) where its value is regarded as known, we would have its value at the point in space of interest (the detector). This would enormously simplify our problem. But since the above flow in xyz - space is very far from incompressible, the above method is totally inapplicable[1]. However, by using, not a single travelling wave, but a <u>statistical ensemble</u> of such waves, randomly out of phase, we can easily establish an equation of continuity in the 5-dimensional "space" of values of the six variables (x, y, z, p_x, p_y, p_z) which satisfy the equation $H = 0$. Now the Hamiltonian theory comes to our aid, showing that this flow in 5-dimensions is <u>incompressible</u>. This is the consequence of Liouville's theorem, of fundamental importance in classical statistical mechanics. Therefore the energy density is constant along each bi-characteristic, or ray in the 5-dimension representation.

The "model" of the action of the ocean in transmitting acoustic energy over long ranges from an emitter of naval interest to a receiver, experiencing all the random viscissitudes of the environment as well as of these two objects, is a <u>statistical ensemble</u> $\{\psi_n\}$ <u>of travelling waves</u> ψ_n, and only approaches a single one (a point source, plane wave, etc.) in the limit.
It happens that for the present purpose it is easier to deal with the ensemble (however near to its limit) than the limit itself.

There is a two-fold situation that may appear paradoxical: First, the sum of two travelling waves is not in general a travelling wave and does not have a wave front in the usual sense. This fact, which should be evident from the analytical expressions, has too frequently been overlooked and has led to errors in some standard text books in acoustics. Second, in spite of the inapplicability of the <u>wave front</u> picture, the <u>ray</u> picture and the Hamiltonian form of the equations continue to be valid. Moreover, the additivity of energies transmitted along several intersecting rays is an immediate consequence of their corresponding to travelling waves that are <u>randomly out of phase</u>. If they were in phase their <u>amplitudes</u> would add vectorially: there would be interference. This is not observed under the physical conditions of our military situation, with its multi-path reception, etc., lending support to the validity of our model.

From the ensemble $\{\psi_n\}$ we are led to the replacement of the E and $\vec{F}$, which were functions of position only, to corresponding quantities which depend on ray direction (momentum) as well. Let (dV) be an element of volume in the xyz-space and let $\vec{D}$ be a direction (a point on the unit sphere). If (dD) is an elementary cone of directions (elementary area on the sphere) containing $\vec{D}$, we shall select the sub-ensemble of $\{\psi_n\}'$ consisting of those waves whose individual wave fronts at (dV) have a direction in (dD). To be precise, this has to be required at some chosen reference point in (dV); but the results depend only infinitesimally on its exact choice. Consider the sum of energies in (dV) contributed by all the members of the sub-ensemble $\{\psi_n\}'$: to quantities of higher order, it is proportional to the volume dV and the area dD, and may be written as

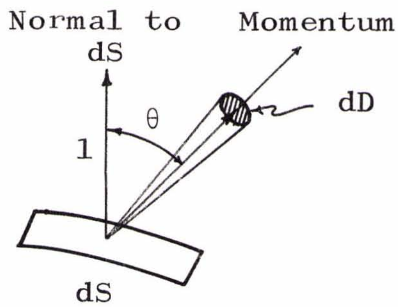
$$E\, dV\, dD = E(x, y, z, p_x, p_y, p_z)\, dV\, dD ,$$

where, of course, p_x, p_y, p_z, are the components of the vector $\vec{D}/c$. The coefficient E in this expression is the energy density per unit 5-volume in the phase-space M_5. When integrated over the whole unit sphere it gives the energy density of the radiation field in ordinary 3-space.

A corresponding definition is given for the energy flux density in M_5. Then by an additive process, starting with the ordinary equations of continuity for the individual elements ψ_n in our ensemble, and making obvious assumptions about the latter, an equation of continuity in the phase space M_5 is obtained. We give only one result at this point: Let dS be an element of surface in ordinary 3-space and dD an element of directions. If θ is the angle between the normal to dS and the direction $\vec{D}$ in dD, the rate of flow of energy across dS due to the waves of ray directions in dD is given by

$$E \cos\theta \; dS \, dD \quad ,$$

where E is the value calculated at a reference point in dS and the direction $\vec{D}$. [Note that any change in $\vec{D}$ and dD during the elementary time $d\tau$ considered has only a higher order effect on the flux.] This result corresponds to the following:



$$[d\Omega^2]/2 = \frac{1}{c^2} \cos\theta \; dS \; dD$$

We shall find an invariantive expression for the $\cos\theta \; dS \, dD/c^2$, which will lead to the desired conclusion of the behaviour of E along each ray.

We return to our Hamiltonian equations. The six first-order differential equations determine one and only one integral curve through each point of the six dimensional phase space of the variables x, y, z, p_x, p_y, p_z. Furthermore, as the independent variable τ increases, each point moves continuously along its integral curve (the ray). This gives rise to a continuous one-parameter group of transformations — a "flow". Liouville's theorem declares that this flow is incompressible.

The only part of the six-dimensional phase space is the sub-space M_5 of five dimensions defined by the equation

$$M : H(x,y,z, p_x, p_y, p_z) = 0 .$$

Since the Hamiltonian H is a first integral it remains constant along each integral curve. Therefore the locus M_5 is invariant in the flow, and contains the whole of every integral curve containing a point in common with it. Thus there is produced a flow in the sub-phase-space M_5. As a corollary of Liouville's theorem it too is conservative.

The concept of integral invariant was developed at the end of the last century by H. Poincaré[2] in connection with dynamical systems, and greatly extended and given wide applications by E. Cartan[3] in the early decades of this century. In its simplest forms it is a familiar notion: the density ρ of a fluid, when integrated over any given volume of the latter, remains invariant during the flow, since the mass it represents is conserved. In a perfect fluid, the integral, around a closed curve drawn in the fluid, of the tangential component of the velocity, is invariant during the flow: this is Lagrange's theorem of the constancy of "circulation", which is basic to his theory, as well as to Helmholtz's theory of vortices. In this case, for invariance, the integral has to be taken about a <u>closed</u> curve: it is called a "relative" integral invariant. When this condition is unnecessary, the invariance is called "absolute". An example of an absolute integral invariant is obtained by applying Stokes' theorem to the circulation, thus expressing the circulation as a surface integral of the normal component of the curl of the velocity — the <u>vorticity</u> (tourbillon).

The Hamiltonian theory gives us a set of integral invariants, starting with the basic relative one. Setting

$$\Omega = p_c \, \delta x + p_y \, \delta y + p_z \, \delta z$$

and using the notation $d\Omega$ for the curl, and bracketed product for "outer" or Grassmann product (actually a determinant operation) we have the series

$$\oint \Omega \, , \quad \int_2 d\Omega \, , \quad \int_4 [d\Omega \, d\Omega] = \int_4 [d\Omega^2] \, ,$$

$$\int_6 [d\Omega^6] \, , \quad \int_5 u \, dS_5 \quad .$$

The existence of the six-dimensional one is simply the statement of Liouville's theorem, and the five-dimensional one is a direct consequence. The "density" u in the latter case can be computed as a simple single-valued expression in terms of the 6-gradient of H.

For our purposes the integrand of the four-dimensional integral invariant is of particular importance because of its simple geometrical interpretation. We have, in fact, when the 4-dimensional element in M_5 is taken as the pair (dS, dD) used before, that

$$[d\Omega^2] = \frac{2}{c^2} \cos\theta \; dS \, dD \quad .$$

This is derived from the expression for the quantity

$$\tfrac{1}{2}[d\Omega^2] = [dp_x \, dx \, dp_y \, dy] + [dp_x \, dx \, dp_z \, dz] + [dp_y \, dy \, dp_z \, dz]$$

where each bracket is the Jacobian determinant of the indicated quantities with respect to the four parameters in the representation of the surface element in question.

Since, as we have seen, the flux of energy across the above element is given by

$$E \, \frac{c^2}{2} \, [d\Omega^2] \quad ,$$

and since as stated before, this flux is the integrand of an integral invariant — by the conservation of energy — it follows by the general theory (actually by Cartan's theorem) that the ratio, namely E, is a scalar invariant, constant along each ray. By referring it back to its point of contact with the emitter, its value can be determined. By doing this for each ray through the receiver, the total energy received can be found by integration. Since one usually assumes the initial values of E at the receiver constant, the above process reduces to that of finding the solid angle subtended at the receiver by the directions of those rays which connect it with the emitter.

The most limited view of the above results is that we have established the validity of a ray tracing process in the case of a general sound speed function $c = c(x,y,z)$ [4]. Actually we have done more, we have laid the basis for a statistical treatment of perturbations of the system. But this cannot profitably be discussed in its general terms in this Conference, so we shall sample it in a simple special case below. Before leaving this subject, it is noted that when our equations are written in general curvilinear coordinates, our densities become multiplied by the factor $\sqrt{g}$, where g is the determinant of the matrix of coefficients in the general expression for the length squared, ds^2. With coordinates appropriate to cylindrical spreading, $\sqrt{g} = r$; for spherical spreading, $\sqrt{g} = r^2 \sin\theta$, etc. This in combination with the invariance of the energy density automatically introduces the appropriate spreading factors, $1/r$, $1/r^2$, etc.

Finally, we note that in our use of integral invariants, they are understood in Cartan's sense — "sliding invariants", remaining constant when all points are slid an arbitrary amount along the integral curves, without requiring to be moved to synchronous points, as required by the Poincaré conception.

We turn now, to illustrate the ideas graphically, to the special case (to which so much of present ray-tracing is confined!) in which we have a "fixed profile", with c depending on depth alone, $c = c(z)$. Then, as is generally known, the equations can be integrated explicitly, requiring, however, a set of numerical integrations of numerically given functions. For purposes of illustration, we shall exhibit the phenomena graphically, after the easy reductions have been made.

The first reduction in this case replaces the geometrical 3-space by a vertical plane, the (x,z) plane or, more appropriately to cylindrical spreading, the (r,z) plane. The two momenta (p_r, p_z) and equation $H = 0$ show that the phase-space becomes a 3-dimensional one, M_3 instead of M_5. This makes it possible to draw diagrams of it on paper.

The next simplification when $c = c(z)$ is the validity of Snell's law in the large, which states that p_r is constant along each ray. This constant, which we denote by w, is called the Snell constant. The set of all rays in M_3 having a given Snell constant are on the locus of the equation

$$p_z^2 + w^2 = 1/c^2 \quad .$$

This makes it convenient to use, for specifying points in M_3, the three coordinates (r,z,p_z). This M_3 is shown in Fig. 1 drawn with these three coordinates as rectangular.

This is not a tank of water — the ocean is represented as the (r,z) plane — but, if a tank of anything, a tank of phase space. But the flow in M_3 is <u>incompressible</u>.

Since the above equation, given the fixed Snell constant value w, does not contain r, its locus is a horizontal cylindrical surface whose elements are parallel to the axis of r. Each ray with this value of w winds around the cylinder in helix-like fashion.

Let a plane be drawn perpendicular to the r-axis. It cuts all the rays in one and only one point. As the value of r at its intersection increases, i.e., as the plane is moved along the r axis and always perpendicular to it, the points of intersection of a given ray move in this plane; such a plane is called the <u>surface of section</u>, and was introduced in the study of dynamical systems by H. Poincaré, and later by G.D. Birkhoff and the author[5]. The transformation induced by the rays when r is changed, as described above, can be pictured as a flow in the plane. During this flow, the representative point of each ray moves about its curve of given Snell constant. Finally, the flow is incompressible. This is a consequence of the sliding integral invariant $\int_{\Sigma} d\Omega$, which, when evaluated on a region of the surface of section, is equal to its area.

Since, as in the more general case, energy flux is a constant along the rays, the application of the relations just outlined can serve as a basis for the study of propagation, showing shadow zones, etc. It may be noted that caustics have disappeared in this representation; they re-appear only when we project sets of rays of M_3 onto the (r,z) plane.

A case of great practical importance is that in which the ranges are very large — quite a number of ray periods. Then the practical uncertainty of the exact lengths of periods leads to the replacement of the energy flux density by its average over a period. Either on this basis, or by reasoning based on "ergodic mixing", we are led to consider an energy flux density, which depends on the Snell constant w only, and therefore has the path curves on the surface of section as its level lines. With a slightly higher degree of perturbing influences, this flux becomes essentially constant over those parts of the surface of section where long-range propagation is not intercepted (by underwater obstructions, etc.) — and zero over the latter parts. Then the acoustic power born by a bundle of rays is proportional to the (invariant) area in which it is cut by the surface of section. On reducing the picture back into geographical space the spreading factor comes out of the equations automatically.

NOTES AND REFERENCES

1. It would be sufficient if, instead of being incompressible, the flow in the x y z-space were conservative, i.e., had a density function whose integral over any piece of this space remains constant: then the ratio of the energy flux density to the latter would be the required first integral. Such a density exists – and is put in evidence by convential ray-tracing – as long as we stay sufficiently close to the emitter. This provides the justification of the standard methods; but only under this proviso. Further away, in fact at distances of particular interest, the densities become increasingly multiple-valued (indeed, singular at the branch-loci, the caustics); therefore the justification breaks down. It is for this reason that the present approach is not submitted merely as an alternative to a more conventional one, but as a method of salvaging the latter when it ceases to be applicable.

2. H. Poincaré, "Méthodes Nouvelles de la Mécanique Célèste", Vol.III (Gauthier-Villard, Paris, 1899).

3. E. Cartan, "Leçons sur les Invariants Intégraux" (Hermann, Paris, 1922).

4. Thus justifying the conventional ray-tracing method in the neighourhood of the emitter, and replacing it by a method that is valid at greater distances. See Ref. 1 above.

5. H. Poincaré, l.c.; G.D.Birkhoff, "The Restricted Problem of Three Bodies" (Rendiconti del Circolo Matematico di Palermo, 23 August 1914); B.O. Koopman, "On Rejection to Infinity and Exterior Motion in the Restricted Problem of Three Bodies", (Trans. Amer. Math Soc., 1927).

6. The simplified picture (given in so many treatments) of the "limiting critical ray" composed of two tangent circles, one above and one below the horizontal ray, is derived from the two-line "approximation" to the acoustic profile. Unfortunately,

for this picture, the differential equations determining the rays involve the derivative of the profile as coefficients. Since the two-line "approximation" has no derivative at the point at which this critical ray is constructed, it is difficult to understand the logic of the construction.

Instances of the surface of section are shown in Fig. 2 (one duct) and Fig. 3 (two ducts). These show the lines of the 2-dimensional flow, along each of which the Snell constant w has a fixed value. They enclose the ducts. In the two duct case, they intersect at the point of maximum sound speed, corresponding to the unstable horizontal ray, approached asymptotically by its neighbours (with increasing or with decreasing range. In Fig. 2 we have heavily shaded the region at which the emitter injects its energy. Since the vertical dimensions of the emitter are small in comparison with the depth, the region is a slender band. Its horizontal extent is wide since this corresponds to the directions (or momentum values) at which it emits power. For a point source, the band would shrink up to a horizontal line segment.

Figure 4 shows the effect of an underwater obstruction, with a key to the calculation on the left, which refers back to the surface of section. Time does not permit us to go into details here; we merely note that the fraction intercepted by the obstruction is the horizontal interval through which the ray (in rz-space) can be moved and still cut the obstruction, divided by the ray period.

Figure 5 shows a graphical method of exhibiting the source-to-duct coupling. The case shown is that of a single duct under an inversion layer, and assumes perfect specular reflection (plus phase randomization) at the water's surface. One may think of the whole diagram as reflection in the latter surface (method of images, or "Lloyd's mirror"), whereupon it resembles the case of three ducts separated by two unstable horizontal rays. Again the thin horizontal band shows the energy injected by the emitter, its heavily shaded part being that portion that survives bottom absorption and can undergo long range propagation. Their ratio might be called the "source-to-duct coupling factor". At the receiver is drawn a horizontal band representing what it can receive (or, by reversal of path, emit). The dotted part of the receiver band shows the part of the emitted energy that reaches the receiver. Clearly the fraction of emitted power reaching the receiver is not even approximately equal to the product of an emitter coupling factor (determined by its depth) times a receiver coupling factor (determined by its depth): given the acoustic profile, the fraction in question is a function of two variables (the two depths); but not a product of two functions of one variable each.

DISCUSSION

The author stated that the theory would have to be reworked for the 5-dimensional case when applied to a range-dependent sound-speed profile. Also, since a flat bottom was assumed in the theory, it would have to be modified if that were not the case.

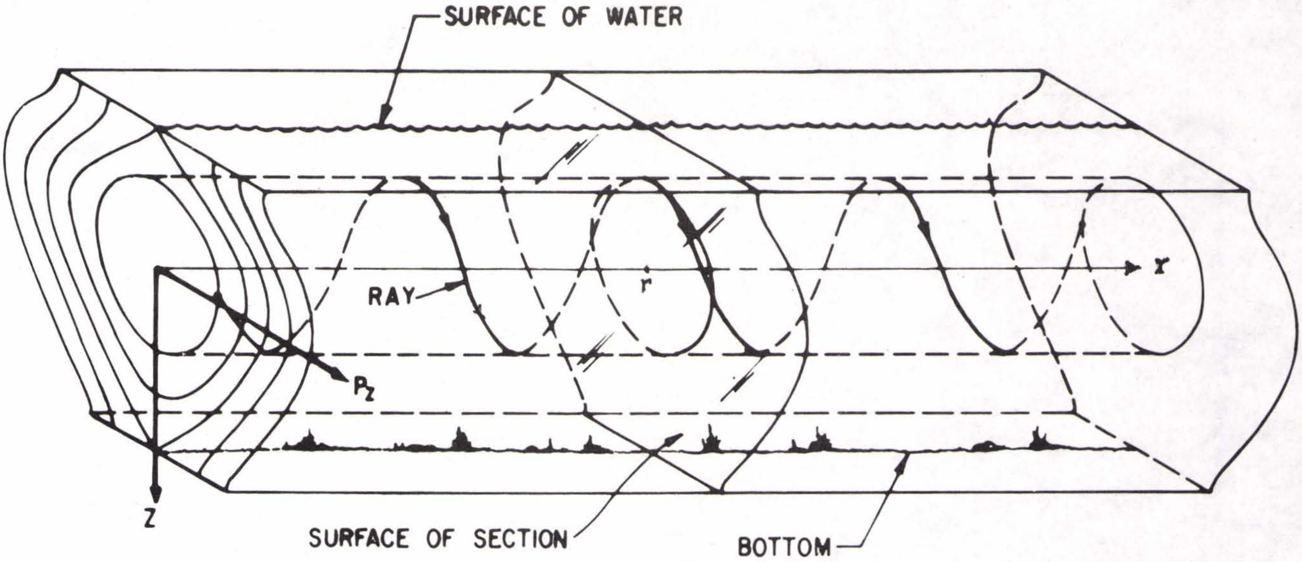

SURFACE OF WATER
RAY
r
x
P_z
Z
SURFACE OF SECTION
BOTTOM

FIG. 1

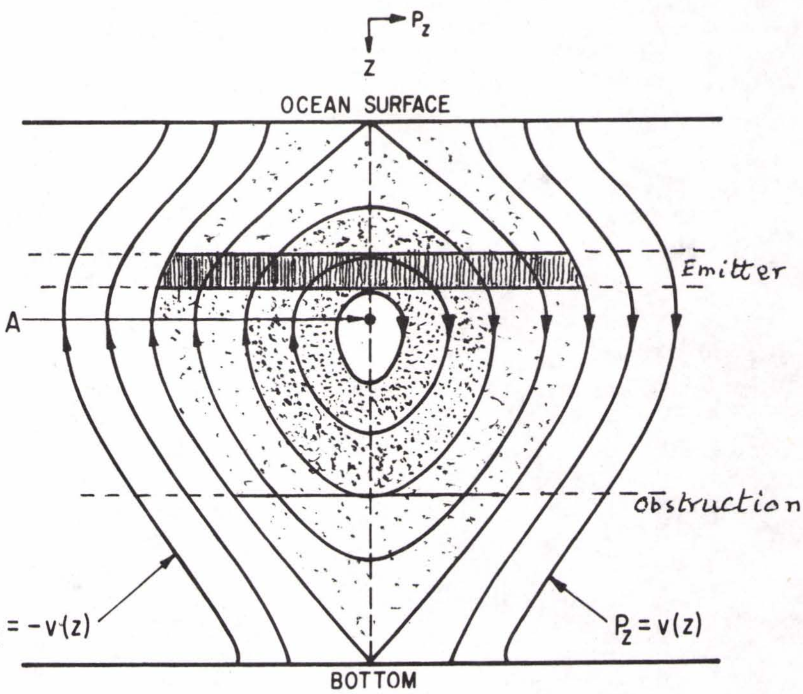

P_z
Z
OCEAN SURFACE
Emitter
A
Obstruction
$P_z = -v(z)$
$P_z = v(z)$
BOTTOM

FIG. 2

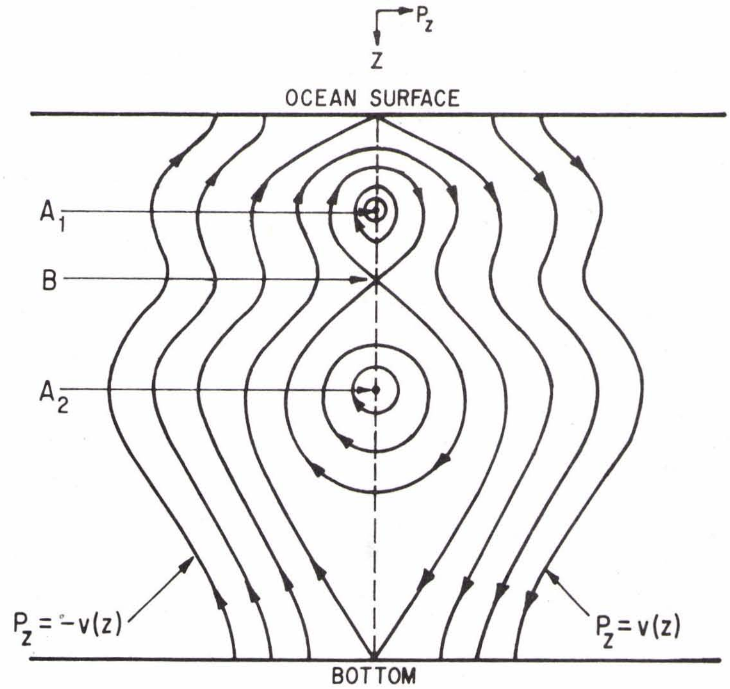


FIG. 3

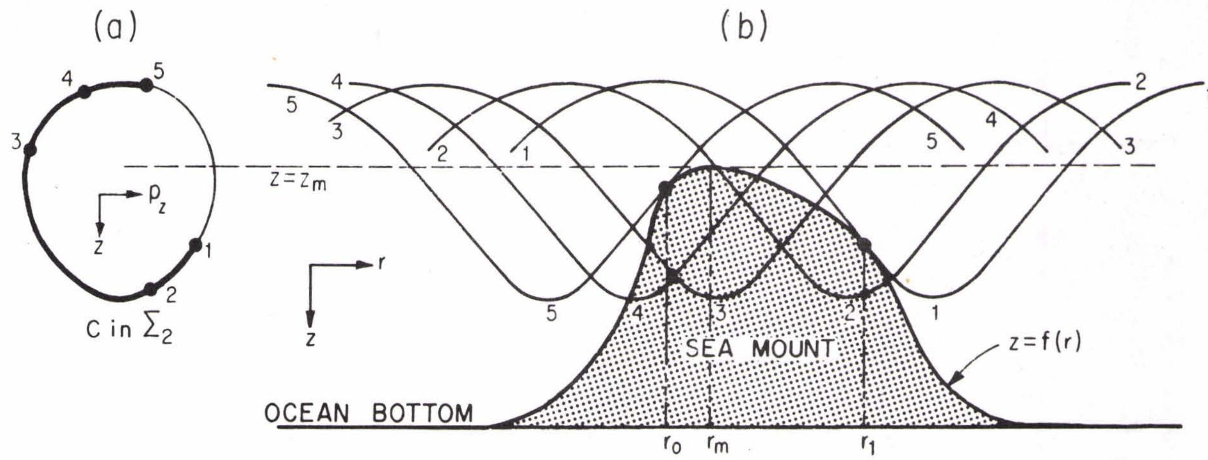


FIG. 4

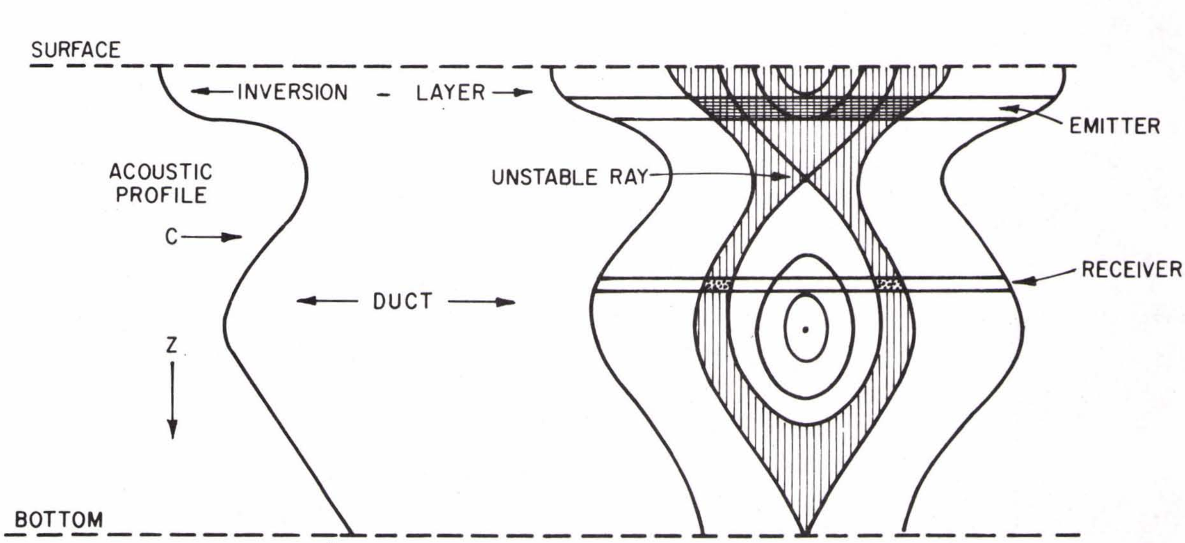


FIG. 5