APPLICATION OF THE RIESZ POTENTIAL TO THE CAUCHY PROBLEM FOR WAVE PROPAGATION IN AN INHOMOGENEOUS MEDIUM

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ABSTRACT

The method of Riesz [Ref. 1] for the solution of hyperbolic partial differential equations is applied to the Cauchy problem for the wave equation. It is shown that the first term in the Riesz potential function, which is represented in series form, yields the geometrical acoustics solution when applied to the problem of radiation from a point source.

THE WAVE EQUATION AND ITS RIEMANNIAN GEOMETRY

We seek a solution to the partial differential equation

$$Lu = \frac{1}{c^2} u_{tt} - \nabla^2 u = f \quad , \qquad \text{[Eq. 1]}$$

where f is a function of $(t,\bar{r}) \in R \times E^3$. For simplicity we shall assume vanishing initial conditions, $u(0,\bar{r}) = u_t(0,\bar{r}) = 0$. The local sound speed c is assumed to be a function of $\bar{r} = (x,y,z)$ only.

Construction of the Riesz potential for the operator L rests on the Riemannian geometry associated with the operator. The semi-Riemannian metric is given by the differential form

$$ds^2 = c^2\, dt^2 - (d\bar{r})^2 \quad . \qquad \text{[Eq. 2]}$$

A displacement for which $ds^2 > 0$ is called timelike; one for which $ds^2 < 0$ is called spacelike. Let $P:(t_0, \bar{r}_0)$ and $Q:(t_1, \bar{r}_1)$ be two points of space time. A geodesic joining P and Q is a curve $\gamma:\{[t(\sigma), r(\sigma)] : 0 \leq \sigma \leq \sigma_0\}$ such that $[t(0), \bar{r}(0)] = (t_0, \bar{r}_0)$, $[t(\sigma_0), \bar{r}(\sigma_0)] = (t_1, \bar{r}_1)$, and

$$s(P,Q) = \int_{\gamma} ds \qquad \text{[Eq. 3]}$$

is an extremum with respect to all curves joining P to Q. Let

$$\left(\frac{ds}{d\sigma}\right)^2 = c^2 \left(\frac{dt}{d\sigma}\right)^2 - \left(\frac{d\bar{r}}{d\sigma}\right)^2 = w \quad .$$

Then the geodesic curves are extremum curves of the integral

$$\int_0^{\sigma_0} w^{\frac{1}{2}} \, d\sigma \quad . \qquad \text{[Eq. 4]}$$

The geodesic curves then satisfy the Euler-Lagrange equations

$$\frac{d}{d\sigma}\left(\frac{\partial w^{\frac{1}{2}}}{\partial v^0}\right) - \frac{\partial w^{\frac{1}{2}}}{\partial t} = 0 \qquad \text{[Eq. 5]}$$

$$\frac{d}{d\sigma}\left(\frac{\partial w^{\frac{1}{2}}}{\partial \bar{v}}\right) - \nabla w^{\frac{1}{2}} = 0 \quad , \qquad \text{[Eq. 6]}$$

where

$$v^0 = \frac{\partial t}{\partial \sigma} \qquad \text{[Eq. 7]}$$

$$\bar{v} = \frac{\partial \bar{r}}{\partial \sigma} \quad . \qquad \text{[Eq. 8]}$$

If we now choose w to be constant along a geodesic (σ is then a linear function of s) the equations for the geodesics become

$$\frac{d}{d\sigma}\left(c^2 v^0\right) = {v^0}^2 c \frac{\partial c}{\partial t} = 0 \qquad \text{[Eq. 9]}$$

$$\frac{d\bar{v}}{d\sigma} = -(v^0)^2 \, c \nabla c \qquad \text{[Eq.10]}$$

Let $w = 1 - \rho^2$. Then $\rho = 1$ is the equation of the characteristic cone through P. We have

$$\left(\frac{ds}{d\sigma}\right)^2 = c^2 v_0^2 - \bar{v}^2 = 1 - \rho^2 \quad . \qquad \text{[Eq. 11]}$$

If $0 \leq \rho \leq 1$ $ds^2 > 0$ and the geodesics are timelike. Since w is constant along a geodesic

$$s = \sigma\sqrt{1 - \rho^2} \quad . \qquad \text{[Eq. 12]}$$

Using the notation introduced by Hadamard we set $\Gamma(P,Q) = s^2(P,Q)$. The region $\{Q: \Gamma(P,Q) \geq 0, \quad t \leq t_1\}$ is called the retrograde conoid with vertex P. We see from Eq. 12 that it is also defined by $0 \leq \rho \leq 1$, $t \leq t_1$. The region $\{Q: \Gamma(P,Q) \geq 0, \quad t \geq t_1\}$ is called the direct conoid with vertex P.

In the theory of the Riesz potential a particular coordinate system for the conoid with vertex P plays an important role. This is the Riemannian coordinate system with coordinates defined by

$$x^i = v^i(0)\sigma, \quad i = 0, 1, 2, 3 \qquad \text{[Eq. 13]}$$

In this coordinate system the geodesics emanating from the point P appear as straight lines. It is shown by Riesz [Ref. 1] that

$$\Delta\Gamma = 8 + 2\sigma \frac{d \ln \sqrt{a}}{d\sigma} \qquad \text{[Eq. 14]}$$

where Δ represents the second order Beltrami operator, or space-time Laplacian, and a is the determinant of the metric tensor, expressed in Riemannian coordinates. We have

$$a = J^2 c^2 \qquad \text{[Eq. 15]}$$

where $J = D\begin{pmatrix} t & x & y & z \\ x^0 & x^1 & x^2 & x^3 \end{pmatrix}$, the Jacobian transformation from the original coordinate system to the Riemannian coordinate system. Expressed in our original coordinates, t, x, y, z, we have

$$\Delta\Gamma = \frac{1}{c}\left(\frac{1}{c}\Gamma_t\right)_t - \frac{1}{c}\nabla\cdot(c\nabla\Gamma)$$

$$= L\Gamma - \nabla \ell n\, c \cdot \nabla\Gamma \qquad \text{[Eq. 16]}$$

$$= L\Gamma + 2\sigma\frac{d\,\ell n\, c}{d\sigma}$$

since $\frac{\partial c}{\partial t} = 0$.

Hence, from Eqs. 14 and 16,

$$L\Gamma = 8 + 2\sigma\frac{d\,\ell n\, J}{d\sigma} \qquad \text{[Eq. 17]}$$

From Eq. 9 we see that $c^2 v^0$ is constant along a geodesic. Letting the constant be $-c_0 = -c(P)$,

$$\frac{dt}{d\sigma} = -\frac{c_0}{c^2} \qquad \text{[Eq. 18]}$$

$$\frac{d\bar{r}}{d\sigma} = \bar{v} \qquad \text{[Eq. 19]}$$

$$\frac{d\bar{v}}{d\sigma} = -\frac{c_0^2}{c^3}\nabla C = \tfrac{1}{2}\nabla\left(\frac{c_0}{c}\right)^2 \qquad \text{[Eq. 20]}$$

Moreover, from Eq. 11,

$$[\bar{v}(0)]^2 = \rho^2 \qquad \text{[Eq. 21]}$$

THE RIESZ POTENTIAL

Following Duff [Ref. 2] we denote by D^p the interior of the retrograde conoid with vertex P. Let S be the intersection of D^p with the initial manifold, $t = 0$, and let D^p_s be the part of the conoid cut-off by the initial manifold, i.e. the intersection of D^p with the half-space $t > 0$. For twice

differentiable functions u, v, defined on D_s^p we have by Green's theorem

$$\int_{D_s^p} (u\,L\,v - v\,L\,u)\,dt\,dx\,dy\,dz$$
$$= \int_{S\cup C_s^p} \{\frac{1}{c^2}\,(uv_t - vu_t)n_t + (v\nabla u - u\nabla v)\cdot\bar{n}\,dS\} \qquad \text{[Eq. 22]}$$

where C_s^p is the part of the characteristic cone $(\Gamma = 0)$ cut off by the initial plane $t = 0$, and $(n_t, \bar{n})$ is the exterior normal to the boundary.

The Riesz potential $V^\alpha(P,Q)$ is defined as a function of points P,Q, in the Riemannian space of the wave equation and the complex variable α. It satisfies the relations

$$LV^{\alpha+2}(P,Q) = V^\alpha(P,Q) \qquad \text{[Eq. 23]}$$

and

$$\lim_{\alpha \to 0} \int_{D_s^p} f(Q)\,V^\alpha(P,Q)\,dt\,dx\,dy\,dz = f(P) \quad , \qquad \text{[Eq. 24]}$$

for any continuous function f. V^α is expressed in the form

$$V^\alpha(P,Q) = \sum_{k=0}^{\infty} \frac{s^{\alpha+2k-4} V_k(P,Q)}{H(\alpha,k)} \quad , \qquad \text{[Eq. 25]}$$

where $s = s(P,Q)$ is the geodesic distance between P and Q. The functions $V_k(P,Q)$ are to be determined from the conditions 23 and 24, while

$$H(\alpha,k) = \pi\,2^{\alpha+k-1}\,\Gamma(\tfrac{1}{2}\alpha)\,\Gamma(\tfrac{1}{2}\alpha+k-1) \qquad \text{[Eq. 26]}$$

For sufficiently large $\mathrm{Re}\,\alpha$, V^α is an analytic function of α and vanishes on C_s^p. Thus, for those values of α, and for functions u satisfying the vanishing initial conditions of the Cauchy problem [Eq. 1], we have from Eq. 22

$$\int_{D_s^p} u\, L V^{\alpha+2}\, dt\, dx\, dy\, dz = \int_{D_s^p} V^{\alpha+2}\, Lu\, dt\, dx\, dy\, dz \quad . \qquad [\text{Eq. } 27]$$

Equation 27 remains valid for all values α to which analytic continuation is possible. From Eqs. 23 and 24, then, letting α tend to zero, we obtain for a solution to Eq. 1

$$u(P) = \lim_{\alpha \to 0} \int_{D_s^p} V^{\alpha+2}\,(P,Q)\, f(Q)\, dt\, dx\, dy\, dz \quad . \qquad [\text{Eq. } 28]$$

Equation 28 provides a representation of the solution to the Cauchy problem. Determination of the coefficients $V_k(P,Q)$ remains.

From the definition given by Eq. 26 and the properties of the Gamma function we obtain the relations.

$$H(\beta,k) = 2(\tfrac{1}{2}\beta + k - 2)\, H(\beta,\ k-1) \qquad [\text{Eq. } 29]$$

$$H(\beta+2,\ k) = 2\beta\,(\tfrac{1}{2}\beta + k - 1)\, H(\beta,k) \quad . \qquad [\text{Eq. } 30]$$

Now

$$L(\Gamma^\beta \psi) = \Gamma^\beta L\psi + \psi L\Gamma^\beta + 4\beta\ \Gamma^{\beta-1}\,\sigma\,\frac{d\psi}{d\sigma} \quad . \qquad [\text{Eq. } 31]$$

[Here we have used Eq.4.5.19 of Duff (Ref. 2).] Using Eq. 17 this becomes

$$L(\Gamma^\beta \psi) = \Gamma^\beta L\psi + 4\beta\Gamma^{\beta-1}\ \{\sigma\,\frac{d\psi}{d\sigma} + (\beta + 1 + \tfrac{1}{2}\sigma\,\frac{d\,\ell n\, J}{d\sigma})\,\psi\ \}. \qquad [\text{Eq. } 32]$$

Operating with L on $V^{\beta+2}(P,Q)$, given by Eq. 25, and using Eq. 32, we have

$$LV^{\alpha+2} = \sum_{k=0}^{\infty} \frac{1}{H(\alpha+2,\,k)}\ \{4(\tfrac{\alpha}{2} + k - 1)\ \Gamma^{\frac{\alpha}{2}+k-2} \qquad [\text{Eq. } 33]$$

$$[\,(\tfrac{\alpha}{2} + k + \tfrac{1}{2}\sigma\,\frac{d\,\ell n\, J}{d\sigma}\,)\ V_k + \sigma\,\frac{dV_k}{d\sigma}] + \Gamma^{\frac{\alpha}{2}+k-1}\, LV_k\ \}$$

Using Eq. 29 this becomes

$$LV^{\alpha+2} = \sum_{k=0}^{\infty} \frac{1}{H(\alpha+2,\,k-1)} \left\{ \left(\alpha + 2k + \sigma \frac{d\,\ell n\, J}{d\sigma}\right) V_k + 2\sigma \frac{dV_k}{d\sigma} + LV_{k-1} \right\} \Gamma^{\frac{\alpha}{2}+k-2} \qquad \text{[Eq. 34]}$$

where we have introduced $V_{-1}(P,Q) = 0$. Now choose $V_k(P,Q)$ so that for $k = 0,1,2,$

$$\left(2k + \sigma \frac{d\,\ell n\, J}{d\sigma}\right) V_k + 2\sigma \frac{dV_k}{d\sigma} + L V_{k-1} = 0 \qquad \text{[Eq. 35]}$$

Then, using Eqs. 29 and 30, we have

$$LV^{\alpha+2} = \sum_{k=0}^{\infty} \frac{\alpha}{H(\alpha+2,\,k-1)} V_k \Gamma^{\frac{\alpha}{2}+k-2} \qquad \text{[Eq. 36]}$$

$$= \sum_{k=0}^{\infty} \frac{1}{H(\alpha,k)} V_k \Gamma^{\frac{\alpha}{2}+k-2} = V_\alpha$$

Thus requiring V_k to satisfy Eq. 35 results in V^α satisfying Eq. 23. We may rewrite Eq. 35 in the form

$$\frac{d}{d\sigma}\left(\sigma^k J^{\frac{1}{2}} V_k\right) = -\tfrac{1}{2}\sigma^{k-1} J^{\frac{1}{2}} LV_{k-1} \qquad \text{[Eq. 37]}$$

Choosing $V_0(0) = c_0$, so that V^α will satisfy Eq. 24 [Ref. 1], we have

$$V_0(P,Q) = C_0 J^{-\frac{1}{2}} \qquad \text{[Eq. 38]}$$

since $J = 1$ at P. For $k \geq 1$ we set $V_k(0) = 0$. Then

$$V_k = -\tfrac{1}{2} J^{-\frac{1}{2}} \sigma^{-k} \int_0^\sigma \sigma^{k-1} J^{\frac{1}{2}} LV_{k-1}\, d\sigma \qquad \text{[Eq. 39]}$$

APPROXIMATE SOLUTION TO THE CAUCHY PROBLEM

Let us now replace $V^{\alpha+2}(P,Q)$ by the first term in the series [Eq. 25] in computing the integral [Eq. 28]. Define

$$\tilde{u}_\alpha(P) = \int_{D_s^p} f(Q) \frac{V_o \Gamma^{\frac{\alpha}{2}-1}}{H(\alpha+2,0)} \, dt\, dx\, dy\, dz \quad . \qquad \text{[Eq. 40]}$$

We consider $\lim_{\sigma \to 0} \tilde{u}_\alpha(P)$ to be a first approximation to the solution to the Cauchy problem. In order to carry out the integration indicated in Eq. 40 we introduce a coordinate system for D_s^p based on the geodesics defined by Eqs. 18, 19, and 20. With $\bar{v} = (v^1, v^2, v^3)$ we set

$$\begin{aligned} v^1(0) &= \rho \cos\theta \cos\varphi \\ v^2(0) &= \rho \cos\theta \sin\varphi \\ v^3(0) &= \rho \sin\theta \end{aligned} \qquad \text{[Eq. 41]}$$

Then Eq. 21 is satisfied. The geodesic equations provide a correspondence between points $(t,x,y,z) \in D_s^p$ and $(\sigma,\rho,\varphi,\theta)$. This correspondence will not be one-to-one in general. If multi-paths occur a point (t,x,y,z) may correspond to many points $(\sigma,\rho,\varphi,\theta)$. However, except at exceptional points (t,x,y,z), each of the points $(\sigma,\rho,\varphi,\theta)$ will have a neighbourhood that is in one-to-one correspondence with a neighbourhood of (t,x,y,z). At the exceptional points, called caustic or focal points, this local one-to-one property will not hold. As a consequence the Jacobian determinant $D\binom{t\ x\ y\ z}{\sigma\ \rho\ \varphi\ \theta}$ will vanish at such points.

The geodesic Eqs. 18, 19 and 20 uniquely define (t,x,y,z) as functions of $(\sigma,\rho,\varphi,\theta)$. Transforming coordinates in Eq. 40 we have

$$\tilde{u}_\alpha(P) = \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\pi/2} d\theta \int_0^1 d\rho \int_0^{\sigma t_o} d\sigma\ f(t,x,y,z)\, V_o \left[\sigma^2(1-\rho^2)\right]^{\frac{\alpha}{2}-1} D\binom{t\,x\,y\,z}{\sigma\,\rho\,\varphi\,\theta}$$

[Eq. 42]

where $\sigma t_o = \sigma t_o(\rho,\varphi,\theta)$ is the value of σ for which the geodesic curve reaches the initial manifold. This will happen at a finite value since we assume c is bounded. It is well known that if $F(\rho,\epsilon)$ is a continuous function for $0 \leq \rho \leq 1$, $\epsilon \geq 0$, that

$$\lim_{\epsilon \to 0} \epsilon \int_0^1 F(\rho,\epsilon)\ (1-\rho)^{\epsilon - 1}\ d\rho \ = \ F(1,0) \qquad \text{[Eq. 43]}$$

We will write, for continous $g(\rho)$,

$$\int_0^1 g(\rho)\ \delta(1-\rho)\ d\rho \ = \ g(1) \quad . \qquad \text{[Eq. 44]}$$

Equation 44 is the definition of the generalized function $\delta(1-\rho)$. Since $H(\alpha+2,\,0) \sim 4\pi/\alpha$ as $\alpha \to 0$ we can apply Eq. 43 to Eq. 42 to obtain

$$\tilde{u}(p) \ = \ \lim_{\alpha \to 0} \tilde{u}_\alpha(P) \qquad \text{[Eq. 45]}$$

$$= \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-\frac{\pi}{2}}^{\pi/2} d\theta \int_0^1 d\rho \int_0^{\sigma t_o} d\sigma\ f V_o\ \sigma^{-2}\ \delta(1-\rho)\ D\begin{pmatrix} t & x & y & z \\ \sigma & \rho & \varphi & \theta \end{pmatrix}$$

Note that in the latter equation we have assumed $f\ V_o\ D\begin{pmatrix} t & x & y & z \\ \sigma & \rho & \varphi & \theta \end{pmatrix}$ to be a continuous function of ρ in a neighbourhood of $\rho = 1$. The solution to the Cauchy problem is represented approximately by Eq. 45. Multipaths cause no problem in this representation, since they are sorted out by the $(\sigma,\rho,\varphi,\theta)$ coordinates.

Letting f represent a point source we set

$$f(t,x,y,z) \ = \ S(t)\ \delta(x,y,z) \qquad \text{[Eq. 46]}$$

where $\delta(x,y,z)$ is the 3-dimensional delta function and $S(t)$ is the transmitted waveform. Although we cannot, strictly speaking, use Eq. 46 in Eq. 45 directly because it is not a continuous function, we could replace the delta function by an approximating sequence of continuous functions. However, the formal manipulations are perhaps more clear if we are less rigorous. Thus we introduce

the generalized function defined by Eq. 46 in Eq. 45. In order to carry out the integration over the delta functions in Eq. 45 it is necessary to revert to (ρ,x,y,z) as variables of integration. Since we do not in general have a one-to-one correspondence between (t,x,y,z) and $(\sigma,\rho,\varphi,\theta)$ a point $(\tau,0,0,0)$ may be covered by many points $(\sigma,\rho,\varphi,\theta)$. We assume there are finitely many. In addition we assume that the origin is isolated from caustics of C_s^p. Each of the points $(\sigma,\rho,\varphi,\theta)$ covering $(\tau,0,0,0)$ then has a neighbourhood U_n that has a one-to-one mapping onto a neighbourhood V_n of $(\tau,0,0,0)$. Then

$$\tilde{u}(P) = \sum_n \int\int_{U_n}\int\int S(t)\,\delta(x,y,z)\,V_o\,\sigma^{-2}\,\delta(1-\rho)\,D\begin{pmatrix} t & x & y & z \\ \sigma & \rho & \varphi & \theta \end{pmatrix} d\sigma\,d\rho\,d\varphi\,d\theta \qquad \text{[Eq. 47]}$$

$$= \frac{1}{4\pi}\int\int_{V_n}\int\int S(t)\,\delta(x,y,z)\,V_o\,\sigma^{-2}\,\delta(1-\rho)\,dt\,dx\,dy\,dz$$

Now, considering t to be a function of (ρ,x,y,z) we write

$$t = t_o - T(\rho,x,y,z) \qquad \text{[Eq. 48]}$$

where T may be interpreted as the travel time along the geodesic emanating from the point P. Then

$$\tilde{u}(P) = -\frac{1}{4\pi}\sum_n \int\int_{V_n'}\int\int S(t_o-T)\,\delta(x,y,z)\,V_o\,\sigma^{-2}\,\delta(1-\rho)\,T_\rho\,d\rho\,dx\,dy\,dz$$

$$= -\frac{1}{4\pi}\sum_n S(t_o-\tau_n)\,V_o\,\sigma_n^{-2}\,T_\rho\,(1,0,0,0) \qquad \text{[Eq. 49]}$$

where τ_n is the travel time along the n-th ray from (x_o,y_o,z_o) to the origin, and σ_n is the corresponding value of σ.

From Eqs. 18, 19 and 20 t and $\bar{r}$ are determined as functions of $(\sigma,\rho,\varphi,\theta)$. We write

$$t = t(\sigma,\rho,\varphi,\theta) \qquad \text{[Eq. 50]}$$

$$\bar{r} = \bar{r}(\sigma,\rho,\varphi,\theta) \qquad \text{[Eq. 51]}$$

Then

$$\frac{\partial t}{\partial \rho} = -\frac{\partial T}{\partial \rho} - \nabla T \cdot \frac{\partial \bar{r}}{\partial \rho} \quad . \qquad \text{[Eq. 52]}$$

Now $\psi = t + T(\rho, x, y, z)$ is, for fixed ρ, an integral surface of the linear partial differential equation

$$\frac{1}{c^2}\,\psi_t^2 - (\nabla\psi)^2 = \text{constant} \qquad \text{[Eq. 53]}$$

whose characteristic strips are generated by the geodesic Eqs. 18, 19 and 20. Comparison with Eq. 11 shows that the constant on the right-hand side of Eq. 53 must be $(1 - \rho^2)/c_o^2$ and $\nabla\psi = \bar{v}/c_o$. Hence

$$c_o \nabla T = \frac{\partial \bar{r}}{\partial \sigma} \quad . \qquad \text{[Eq. 54]}$$

Thus, from Eq. 52

$$\begin{aligned} -c_o \frac{\partial T}{\partial \rho} &= c_o \frac{\partial t}{\partial \rho} + \frac{\partial \bar{r}}{\partial \rho} \cdot \frac{\partial \bar{r}}{\partial \sigma} \\ &= c_o \frac{\partial t}{\partial \rho} + \frac{\partial \bar{r}}{\partial \rho} \cdot \bar{v} \quad . \end{aligned} \qquad \text{[Eq. 55]}$$

Now, differentiating Eq. 55 with respect to σ we have

$$\frac{\partial}{\partial \sigma}\left(-c_o \frac{\partial T}{\partial \rho}\right) = c_o \frac{\partial^2 t}{\partial \rho\, \partial \sigma} + \frac{\partial^2 \bar{r}}{\partial \rho\, \partial \sigma} \cdot \bar{v} + \frac{\partial \bar{r}}{\partial \sigma} \cdot \frac{\partial \bar{v}}{\partial \sigma} \quad . \qquad \text{[Eq. 56]}$$

Let us temporarily write $(c_o/c)^2 = \beta(x, y, z)$. Then the geodesic Eqs. 18, 19 and 20, become

$$c_o \frac{\partial t}{\partial \sigma} = \beta \qquad \text{[Eq. 57]}$$

$$\frac{\partial \bar{v}}{\partial \sigma} = \tfrac{1}{2} \nabla\beta \quad . \qquad \text{[Eq. 58]}$$

From Eq. 11 we obtain

$$\beta - \bar{v}^2 = 1 - \rho^2$$

or

$$\bar{v}^2 = \beta + \rho^2 - 1 \quad . \qquad \text{[Eq. 59]}$$

Referring again to Eq. 56 we then have

$$\frac{\partial}{\partial\sigma}\left(-c_o \frac{\partial T}{\partial\rho}\right) = -\frac{\partial\beta}{\partial\rho} + \frac{1}{2}\frac{\partial}{\partial\rho}(\bar{v}^2) + \frac{\partial\bar{r}}{\partial\rho}\cdot\left(\frac{1}{2}\nabla\beta\right) = \rho \qquad \text{[Eq. 60]}$$

Hence

$$-c_o \frac{\partial T}{\partial\rho} = \rho\sigma \qquad \text{[Eq. 61]}$$

since the relation clearly is valid for small σ. Thus Eq. 49 becomes

$$\tilde{u}(P) = -\frac{1}{4\pi}\sum_n S(t_o - \tau_n)\, V_o/(C_o\, \sigma_n) \qquad \text{[Eq. 62]}$$

where V_o is given by Eq. 38.

The Riemann coordinates involved in the definition of J [Eq. 15] are given in terms of $(\sigma,\rho,\varphi,\theta)$ by

$$x^0 = -\sigma/c_o \qquad \text{[Eq. 63]}$$

$$x^1 = \rho\sigma\cos\theta\cos\varphi \qquad \text{[Eq. 64]}$$

$$x^2 = \rho\sigma\cos\theta\sin\varphi \qquad \text{[Eq. 65]}$$

$$x^3 = \rho\sigma\sin\theta \qquad \text{[Eq. 66]}$$

Then

$$J = D\begin{pmatrix} t & x & y & z \\ x^0 & x^1 & x^2 & x^3 \end{pmatrix}$$

$$= D\begin{pmatrix} t & x & y & z \\ \sigma & \rho & \varphi & \theta \end{pmatrix} / D\begin{pmatrix} x^0 & x^1 & x^2 & x^3 \\ \sigma & \rho & \varphi & \theta \end{pmatrix} \qquad \text{[Eq. 67]}$$

$$= -D\begin{pmatrix} t & x & y & z \\ \sigma & \rho & \varphi & \theta \end{pmatrix} \cdot c_o/(\rho^2 \sigma^3 \cos\theta)$$

$$= -c_o T_\rho D\begin{pmatrix} x & y & z \\ \sigma & \varphi & \theta \end{pmatrix} /(\rho^2 \sigma^3 \cos\theta) = D\begin{pmatrix} x & y & z \\ \sigma & \varphi & \theta \end{pmatrix} / (\rho\sigma^2 \cos\theta)$$

Hence, setting $\rho = 1$,

$$V_o = c_o \sigma \left[\cos\theta / D\begin{pmatrix} x & y & z \\ \sigma & \varphi & \theta \end{pmatrix}\right]^{\frac{1}{2}} \qquad \text{[Eq. 68]}$$

and Eq. 62 becomes

$$\tilde{u}(P) = \frac{1}{4\pi} \sum_n S(t_o - \tau_n) \left[\cos\theta / D\begin{pmatrix} x & y & z \\ \sigma & \varphi & \theta \end{pmatrix}\right]^{\frac{1}{2}} \qquad \text{[Eq. 69]}$$

This is the geometrical acoustics solution in its generalized form.

REFERENCES

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DISCUSSION

The author confirmed that the signal distortion can be obtained directly if the source function is bounded.