SENSITIVITY OF RAY THEORY TO INPUT DATA

by

J.L. Reeves
Department of the Navy
Naval Ship Systems Command
Sonar Research Directorate
Washington, D.C., U.S.

and

L.P. Solomon
Tetra Tech. Inc.
Arlington, Va., U.S.

ABSTRACT

Acoustic propagation problems which are solved using ray theory require certain input data. In particular, the source and receiver locations, sound-velocity field and bottom and surface conditions are required. Frequently, this data is known only to within certain confidence limits. Ray calculations are performed assuming that the input data is known to the required accuracy. A theory is presented which indicates the sensitivity of the calculations to small variations in the input data. The ray equation is characterized by a second-order ordinary differential equation; the intensity can be calculated along the particular ray directly. The necessity of having twice continuously differentiable velocity, surface and bottom profiles is clearly demonstrated in the theory. Specific examples are provided for specialized velocity profiles.

INTRODUCTION

The entire field of geometrical acoustics is expressed mathematically by the solutions associated with the first-order term in the asymptotic expansion of the wave equation, where the expansion parameter is proportional to the inverse powers of the wavenumber, k. Inherent in any expansion procedure is an error analysis which will describe the range of applicability for the expansion. Ray theory provides an accurate approximation to the exact solutions of the wave equation so long as:

$$\frac{\delta\left(\frac{dc}{dz}\right)}{c/\lambda} << 1 \quad . \qquad \text{[Eq. 1]}$$

This expression indicates that the solution of the eikonal equation will be an excellent approximation to the wave equation if the fractional change in the velocity gradient over a wavelength is small compared with 1. It is clear that this condition will be satisfied if the frequencies are sufficiently high and, for present acoustic systems operating throughout the world, this condition is met with ease. Other problems can arise in using an approximate theory which are not themselves dependent upon the analytical techniques required to derive the theory; namely, the possible sensitivity which the theory has to errors in input data or boundary conditions. Any problem in underwater acoustic propagation requires a speed of sound curve, or velocity profile, as input. This velocity profile may be simple or complicated; the ray paths will behave accordingly. The question which must be posed and, in fact, which we have addressed ourselves to is: if there are errors in the description of the velocity profile, how will these errors affect the original solutions? Extremely simple models, which are characterized by linear differential equations, generally can be considered to have the following characteristics: small changes in the model or in the inputs will produce small changes in the results. With the more complicated models which are now available in underwater acoustics, this is not necessarily the case.

Ray theory requires the specification of a source and receiver point as well as the specification of the initial ray angle, θ_0. This angle may only be known to within certain limits. This will clearly show up under both long and short ranges as an error in the transmission loss, phase, ray path, etc. It is our purpose to present here a simplified theory which will allow the calculation of errors due to input conditions given a technique to calculate the original "error free" solutions.

Section I will deal with various types of velocity profiles. Section II will present a technique to obtain the necessary information to generate the data analysis. Section III will develop a general model for calculating the sensitivity of the original error free model to variations in input data. Section IV will present the conclusions.

I. DISCUSSION OF VELOCITY PROFILES

Present models incorporate a number of analytical techniques for describing velocity profiles. In general, the use of certain of these functions gives rise to at least one engaging characteristic — the ray paths connected with the velocity profiles can be calculated from analytical functions. Specifically, velocity profiles have been described using the following techniques:

Constant Velocity

If the velocity is assumed to be constant, then the ray paths are straight lines. Errors could arise by assuming that the velocity is a different constant than originally specified. The time delay, or phase, will be in error, but the straight lines which are, in fact, the ray paths are dependent only upon the initial angle and source position and, therefore, are not dependent upon the numerical value which is selected. However, there can be errors introduced by incorrectly assigning values for source and receiver depths as well as initial ray angle.

Linear Velocity Profile

Linear velocity profiles lead to ray paths which are circular arcs. There are two errors which can arise in describing linear velocity profiles. They are, of course, the slope and the intercept. The ray paths will again be in error due to the error in initial ray angle and source placement. Phase arrival times will be in error.

Higher Order Velocity Profiles

Curvilinear velocity profiles can be characterized by hyperbolic cosines, parabolas and the Epstein profile, to name a few. All these profiles have a number of parameters which enable them to fit actual data; for example, the Epstein profile is a five-parameter profile each of which can be adjusted to give the best fit to actual data. Once these parameters are specified, then the ray paths are also known, since they are a function of the parameters of the velocity profile [Refs. 1 & 2].

It is clear that all the above-mentioned cases have certain difficulties associated with them. Generally, they are incapable of predicting certain variations in the velocity profile. For example, the Epstein profile has difficulty predicting surface ducts. Surface ducts are, of course, characterized by a local maximum in the velocity profile some distance beneath the surface. Another difficulty is that they are merely a best fit to the data and do not easily provide a technique for generating an error analysis.

Splines

Splines are a well known mathematical technique for generating smooth curves to fit data points. The technique is simply described by Ahlberg et al [Ref. 3]. The simplest non-trivial spline is a cubic. The technique is to assume (for example) that a cubic polynomial with unspecified coefficients would describe the curve between two data points. Another cubic polynomial with unspecified coefficients would be employed between the next two data points. At the common data point, the function, its first and second derivatives are all required to be continuous. These conditions of continuity will be almost enough to provide a unique determination of the coefficients of the various polynomials. If there are N data points, then there would be N-1 cubic polynomials.

There are two conditions which have yet to be met and these are at the end points. Generally, it is assumed that the second derivatives outside the range of interest is zero. These two conditions will then be sufficient to determine all the coefficients. This is not a unique way of specifying the coefficients, but it will serve for purposes of example. A good property of splines is that they will allow for matching all input velocity points exactly. There is no "best fit". The drawbacks are unfortunate in that cubic polynomials lead to elliptic integrals which are not solvable in closed form.

All velocity profiles discussed already have been assumed to be functions of depth only. If it is required that the velocity structure be represented as a function of depth and range, then the only technique which is readily applicable is the spline function. The theory of two-dimensional splines is well known and understood and, in fact, is directly applicable to this problem.

II. DIFFERENTIAL EQUATIONS FOR THE MODELS

Ray Paths

A description of the ray paths comes from the solution of the eikonal equation. In fact, the ray paths are the trace of the normal to the wave front as it proceeds through the medium. Since the eikonal equation is difficult to solve, another formulation has been employed: Fermat's Principle of Least Time. This leads directly to Euler's equation which implies that the extremals will satisfy a particular differential equation.

$$\frac{d}{dx}\left[\frac{f(x,z)\,z'}{\sqrt{1+z'^2}}\right] = \frac{\partial f}{\partial z}\sqrt{1+z'^2} \qquad \text{[Eq. 2]}$$

where $f(x,z)$ is given by:

$$f(x,z) = \frac{1}{c(x,z)} \qquad \text{[Eq. 3]}$$

where $c(x,z)$ is the velocity field.

Carrying out the indicated operations leads to:

$$z'' = \frac{1+z'^2}{f}\,(f_z - z'f_x) \; . \qquad \text{[Eq. 4]}$$

The subscript notation for partial derivatives is employed, viz.,

$$\frac{\partial f}{\partial z} = f_z, \text{ etc.}$$

The initial conditions for the path are:

$$\begin{aligned} z(0;\, z_0,\, \theta_0) &= z_0 \\ z'(0;\, z_0,\, \theta_0) &= \tan\theta_0 \end{aligned} \qquad \text{[Eq. 5]}$$

where z_0 and θ_0 are the source depth and initial ray angle, respectively. This formulation is described in detail by Solomon and Armijo [Ref. 4]. Arc length and travel time can be calculated through quadrature.

Intensity

Calculation of the intensity is, in most cases, of primary interest in acoustic propagation problems. In general, if variation in intensity is assumed to be only due to geometrical spreading, then the intensity is given by:

$$\frac{I(x)}{I_0} = \frac{1}{x} \cdot \frac{\cos \theta_0}{\cos \theta} \cdot \frac{1}{\partial z/\partial \theta_0} \qquad \text{[Eq. 6]}$$

where x is the range and θ is the local ray angle measured with respect to the horizontal.

The difficulty in calculating the intensity is easily seen to be the calculation of the function $\partial z/\partial \theta_0$.

If the ray paths are represented by simple functions, then this derivative may be calculated directly. In general, however, the problems are many and many models approximate this function at any range by using:

$$\frac{\partial z}{\partial \theta_0} \approx \frac{\delta z}{\delta \theta_0} \qquad \text{[Eq. 7]}$$

where δz is the difference $z(x;z_0, \theta_0 + \delta\theta_0) - z(x;z_0, \theta_0)$ calculated from both ray paths at a particular range. This technique is numerically acceptable if the two rays remain in close proximity to one another and if roundoff errors are negligible. Unfortunately, neither requirement is true for either long range calculations or rays that reflect many times from the bottom. Furthermore, the technique is undesirable because it requires the calculation of an extra ray path for each ray which is traced to a neighbourhood of the receiver.

Solomon and Armijo [Ref. 4] have demonstrated that it is a simple task to calculate the intensity directly along a ray path. In order to do so, they show that it is necessary to calculate the partial derivative $\partial z/\partial \theta_0$.

They define the function:

$$\zeta(x;z_0, \theta_0) = \frac{\partial}{\partial \theta_0} z(x;z_0, \theta_0) \quad . \qquad \text{[Eq. 8]}$$

Differentiating the ray path equation with respect to the parameter θ_0 and making the appropriate substitutions yields the second order linear differential equation for ζ:

$$\zeta'' = \left[\frac{2z'z''}{1+z'^2} - (1+z'^2)\frac{f_x}{f}\right]\zeta' + \left[\frac{1+z'^2}{f}(f_{zz} - z'f_{xz}) - z''\frac{f_z}{f}\right]\zeta \,. \qquad \text{[Eq. 9]}$$

The initial conditions for Eq. 9 are:

$$\begin{aligned} \zeta(0;z_0,\theta_0) &= 0 \\ \zeta'(0;z_0,\theta_0) &= \sec^2\theta_0 \,. \end{aligned} \qquad \text{[Eq. 10]}$$

The calculation of intensity is now reduced to the problem of solving Eq. 4 with the initial conditions of Eq. 5. The numerical solution of Eq. 4 is certainly no more difficult than tracing an additional ray and the needless approximation of Eq. 7 has been removed. The numerical procedures for solving systems of ordinary differential equations may now be applied to the simultaneous determination of both z and ζ from the differential Eqs. 4 and 9. Since both differential equations are second order, the combined problem is equivalent to a system of four first order differential equations. After employing one of the several available numerical techniques to solve this system, the intensity may be calculated anywhere along the ray path from Eqs. 6 and 8.

IV. SENSITIVITY TO INPUT DATA

Inputs

All acoustic propagation models are obliged to accept certain input data. This model assumes that mean data is provided with some estimate of error as well. For example, the input with some tolerance could be provided. The model requires, therefore, the following input data:

1. Velocity as a function of depth (and conceivably range).
2. Input angles.
3. Source and receiver positions.
4. Bottom profile (as well as reflection coefficients).

General Theory

The ray paths are assumed to be described by:

$$z'' = \frac{(1+z'^2)}{f} [f_z - z'f_x] \qquad \text{[Eq. 11]}$$

where

$$f(z,x) = \frac{1}{c(z,x)} \quad . \qquad \text{[Eq. 12]}$$

Assume that the ambient solution is $z_0(x)$ and this is related to the ambient speed of sound velocity profile $f_0(z,x)$.

Further, let us assume that

$$f(z,x) = f_0(z,x)\,[1+\varepsilon(z,x)] \ . \qquad \text{[Eq. 13]}$$

The error function will be related to the error in $z_0(x)$. That is:

$$z(x) = z_0(x) + \Delta(x) \ . \qquad \text{[Eq. 14]}$$

The boundary conditions are at $x=0$

$$\begin{aligned} z &= z_0, \Delta = 0 \\ z' &= \tan\theta_0 = z_0', \ \Delta' = 0 \ . \end{aligned} \qquad \text{[Eq. 15]}$$

This is where initial angle error arrives. Clearly $z_0(x)$ satisfies

$$z_0'' = \frac{1+z_0'^2}{f_0} [f_{oz} - z_0' f_{ox}] \ . \qquad \text{[Eq. 16]}$$

Clearly, since $z(x)$ satisfies Eq. 11 and $z_o(x)$ satisfies Eq. 16, an equation for $\Delta(x)$ must satisfy some relationship. Substitution of Eq. 14 into Eq. 11 and making use of Eq. 16 and assuming that a linearization procedure is possible, the relationship which $\Delta(x)$ must satisfy is:

$$\Delta'' = -\Delta'\{(1+z_o'^2)\frac{f_{ox}}{f_o} + \frac{2z_o'}{f_o}(f_{oz} - z_o' f_{ox})\} + (\varepsilon_z - z_o'\varepsilon_x)(1+z_o'^2)\,. \qquad \text{[Eq. 17]}$$

A similar expansion can be provided for the intensity.

It will be recalled that ζ (and ζ_o) satisfy

$$\zeta' = [\frac{2z'z''}{1+z'^2} - (1+z'^2)\frac{fx}{f}]\,\zeta' + [\frac{1+z'^2}{f}(f_{zz} - z'f_{xz}) - \frac{z''f_z}{f}]\,\zeta \qquad \text{[Eq. 18]}$$

with the conditions

$$\begin{aligned} \zeta(0) &= 0 \\ \zeta'(0) &= \sec^2\theta_o \end{aligned} \qquad \text{[Eq. 19]}$$

similarly to the perturbation expansion for $z(x)$, we will assume that $\zeta(x)$ may be written as

$$\zeta(x) = \zeta_o(x) + \varkappa(x) \qquad \text{[Eq. 20]}$$

substitution of Eqs. 20, 13 and 14, and expanding, keeping only first order terms gives rise to

$$\varkappa'' = A_o\varkappa' + B_o\varkappa + A_1\zeta_o' + B_1\zeta_o \qquad \text{[Eq. 21]}$$

where A_o, B_o, A_1, B_1 are given by

$$A_o = \frac{2z_o'z_o''}{1+z_o'^2} - (1+z_o'^2)\frac{f_{ox}}{f_o} \qquad \text{[Eq. 22a]}$$

$$B_o = (1+z_o'^2)[\frac{f_{ozz} - z_o' f_{oxz}}{f_o}] - \frac{z_o'' f_{oz}}{f_o} \qquad \text{[Eq. 22b]}$$

$$A_1 = \frac{2}{1+z_o'^2}\left[\frac{(1-z_o'^2)\, z_o''\,\Delta'}{(1+z_o'^2)} + z_o'\,\Delta''\right] - \frac{2\Delta' z_o' f_{ox}}{f_o} - \epsilon_x(1+z_o'^2)\,. \qquad \text{[Eq. 22c]}$$

$$B_1 = \epsilon_z\,\frac{(1+z_o'^2)}{f_o}\cdot\left[(2f_{oz} - z'f_{ox}) - z_o''\right]$$
$$- \epsilon_x\; z_o'(1+z'^2)\frac{f_{oz}}{f_o} - \Delta'\left[(1+3z_o'^2)\frac{f_{oxz}}{f_o} - 2z_o'\frac{f_{ozz}}{f_o}\right]$$
$$+ \epsilon_{zz}\left[1+z_o'^2\right] - \epsilon_{xz}\left[z_o'(1+z_o'^2)\right] - \Delta''\frac{f_{oz}}{f_o}\,. \qquad \text{[Eq. 22d]}$$

It is clear that $\varkappa(x)$ satisfies an ordinary differential equation which is non-homogeneous. The non-homogeneous term or forcing function is dependent upon the ray path and its perturbations, and the speed of sound and its perturbations. It is further seen that finding exact solutions to Eq. 21 are extremely difficult. Let us now investigate certain special cases.

Special Cases

Case 1:

Assume

$$f_o = \text{constant}\,. \qquad \text{[Eq. 23]}$$

Then:

$$z_o(x) = x\tan\theta_o \qquad \text{[Eq. 24]}$$

$$\Delta(0) = \Delta'(0) = 0\,. \qquad \text{[Eq. 25]}$$

Let us further assume that there is no error in x, thus,

$$\epsilon_x = 0\,. \qquad \text{[Eq. 26]}$$

And assume that the error in depth is given by:

$$\epsilon = \alpha z\,. \qquad \text{[Eq. 27]}$$

This leads to

$$\Delta(x) = \alpha\cdot\sec^2\theta_o\frac{x^2}{2}\,. \qquad \text{[Eq. 28]}$$

The appropriate boundary conditions are at $x = 0$

$$z_0(0) = z_0$$
$$z(0) = z_0 + \Delta_0 \qquad \text{[Eq. 29]}$$
$$\Delta(0) = \Delta_0$$

The error in initial angle we represent as:

$$\theta(0) = \theta_0 + \chi$$
$$z_0'(0) = \tan\theta_0 \ . \qquad \text{[Eq. 30]}$$

This corresponds to:

$$z'(0) = z_0'(0) + \Delta'(0) \ . \qquad \text{[Eq. 31]}$$

Further,

$$\tan(\theta_0 + \chi) = \tan\theta_0 + \Delta'(0), \qquad \text{[Eq. 32]}$$

which leads to:

$$\Delta'(0) = \Delta_0' = \frac{\tan\chi\,(1 - \tan^2\theta_0)}{1 - \tan\theta_0 \tan\chi} \ . \qquad \text{[Eq. 33]}$$

Thus, the correct solution for the error $\Delta(x)$ including errors in initial position and ray angle as well as velocity profile error, is given by:

$$\Delta(x) = \Delta_0 + \Delta_0' x + \frac{\alpha x^2}{2} \sec^2\theta_0 \ . \qquad \text{[Eq. 34]}$$

Utilization of Eq. 21 will allow us to calculate error in calculating the intensity. Recall that

$$\zeta_0 = \frac{\partial z_0}{\partial \theta_0} = x \sec^2\theta_0 \ . \qquad \text{[Eq. 35]}$$

Then,

$$\varkappa'' = 2 \sin\theta_0 \sec^7\theta_0 \ . \qquad \text{[Eq. 36]}$$

The boundary conditions on $\varkappa(x)$ are:

$$\varkappa(0) = 0$$

$$\varkappa'(0) = \sec^2(\theta_0 + \chi) - \sec^2\theta_0 = \varkappa_0' \,. \qquad \text{[Eq. 37]}$$

Thus

$$\varkappa(x) = \varkappa_0' x + \alpha x^2 \sin\theta_0 \sec^7\theta_0 \,. \qquad \text{[Eq. 38]}$$

Case 2:

Assume that

$$f_0(x, z) = f_0(z) \,. \qquad \text{[Eq. 39]}$$

Then $z_0(x)$ satisfies:

$$z_0''(x) = (1 + z_0'^2)\,\frac{f_{oz}}{f} \,. \qquad \text{[Eq. 40]}$$

If we let $f_0(z)$ be given by

$$f_0(z) = (\alpha z)^{-1} \qquad \text{[Eq. 41]}$$

then the paths are circular arcs.

In particular, $z_0(x)$ is given by:

$$z_0^2(x) = z_0^2(0)\sec^2\theta_0 - (x + z_0(0)\tan\theta_0)^2 \,. \qquad \text{[Eq. 42]}$$

The error in the ray path $\Delta(x)$ must then satisfy

$$\Delta''(x) = -\Delta'(x)[a_1 + 2a_2 x] + [\epsilon_z - z_0'\epsilon_x][1 + z_0'^2] \qquad \text{[Eq. 43]}$$

where,

$$a_0 = z_0^2(0), \qquad a_1 = -2z_0(0)\tan\theta_0, \qquad a_2 = -1 \,. \qquad \text{[Eq. 44]}$$

The homogeneous solution to Eq. 43 is:

$$\Delta_H(x) = \frac{\sqrt{\pi}}{2}\,\frac{\Delta_H'(0)}{\alpha}\,\mathrm{erf}(\alpha x) + \Delta_H(0) \,. \qquad \text{[Eq. 45]}$$

And the total solution is given by:

$$\Delta(x) = \frac{\sqrt{\pi}}{2} \frac{\Delta'_H(0)}{\alpha} \quad \mathrm{erf}(\alpha x) + \Delta_{particular} \quad . \qquad \text{[Eq. 46]}$$

Let us assume that:

$$\varepsilon(z, x) = \varepsilon(z) \quad . \qquad \text{[Eq. 47]}$$

This assumption leads to:

$$\varepsilon_x = 0 \quad . \qquad \text{[Eq. 48]}$$

Furthermore, assume that

$$\varepsilon_z = \text{constant} = \beta \quad . \qquad \text{[Eq. 49]}$$

The particular solution is in the form:

$$\Delta_{particular} = x^2 \sum b_n x^n \quad . \qquad \text{[Eq. 50]}$$

The recursion relationship for the coefficients is:

$$b_{\varkappa+2}(\varkappa+4)(\varkappa+3) + a_1 b_{\varkappa+1}(\varkappa+3) + 2a_2 b_\varkappa(\varkappa+2) = 0 \quad . \qquad \text{[Eq. 51]}$$

In particular, the first few coefficients are given:

$$b_0 = \beta\widetilde{A}_0/2 \qquad \text{[Eq. 52a]}$$

$$b_1 = \frac{\beta[\widetilde{A}_1 - a_1\widetilde{A}_0]}{6} \qquad \text{[Eq. 52b]}$$

$$b_2 = \frac{\beta\widetilde{A}_2 - \frac{a_1}{2}\beta(\widetilde{A}_1 - a_1\widetilde{A}_0) - 2a_2\beta\widetilde{A}_0}{12} \qquad \text{[Eq. 52c]}$$

where:

$$\begin{aligned} \widetilde{A}_0 &= 4a_0 + a_1^2 \\ \widetilde{A}_1 &= 4a_1 + 4a_2 a_1 \\ \widetilde{A}_2 &= 4a_2(1 + a_2) \quad . \end{aligned} \qquad \text{[Eq. 53]}$$

Thus, the complete solution for $\Delta(x)$ is given by:

$$\Delta(x) \frac{\sqrt{\pi}\Delta'(0)}{2\alpha} \operatorname{erf}(\alpha x) + \Delta(0) + x^2 \sum b_n x^n \ . \qquad \text{[Eq. 54]}$$

To sum up, we have solved the special case where:

$$\epsilon(z) = \beta z \qquad \Delta(0) = \Delta_0$$

$$f_0(z) = (\alpha z)^{-1} \qquad \Delta'(0) = \frac{\chi(1 - \tan^2 \theta_0)}{1 - \chi \tan \theta_0} \ . \qquad \text{[Eq. 55]}$$

The limiting cases lead to:

$$\alpha x \to \infty: \ \Delta(x) \to \frac{\Delta'(0)}{\alpha} + \Delta(0) + x^2 \sum b_n x^n \qquad \text{[Eq. 56]}$$

$$\alpha x \to 0: \ \Delta(x) \to \Delta'(0)\, x + \Delta(0) + x^2 \sum b_n x^n \ . \qquad \text{[Eq. 57]}$$

The procedure for calculating $\varkappa(x)$ is straightforward, but rather tedious.

V. CONCLUSIONS

This model is presented as a simple procedure for attempting to understand the effects of error in input data on ray path calculations. It is quite clear that these effects are not necessarily small nor can they be considered irrelevant. It is suggested that this formulation of the acoustic propagation model has within it the seeds for a full-scale deterministic model which can then predict in a uniquely defined manner the problems and effects of errors inputs. Furthermore, this technique may be of utility in attempting to analyse a deterministic-stochastic model where the error inputs are provided in a statistical manner. It seems clear that this model would be applicable if the means of the individual inputs could be considered to be deterministic by utilizing our techniques and then using the model; a probability density function could be derived by considering the interaction of the probability functions of the input.

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DISCUSSION

The second author stated that this perturbation method avoided the necessity of recalculating the ray path for each perturbation. In this way it conformed to standard perturbation techniques. In reply to a further question regarding whether the effect of perturbation was more marked in areas of low sound speed gradient, he remarked that as these methods had not been programmed yet the answer to this and similar general queries was still unknown.