SPECIAL FORMULATION OF MODIFIED RAY ANALYSIS FOR MACHINE COMPUTATION

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When the reflection coefficient for waves incident on a refracting or bounded region has a phase $\varkappa(\theta_0)$ that is a function of the angle of incidence θ_0 (measured at some reference level $Z = Z_0$), then in consequence, individual rays, or "beams" will be displaced from the location predicted by Snell's law. [See, for example, Ref.1, Ch. I). By $\varkappa(\theta_0)$ we mean phase changes other than those corresponding to the geometrical-optics phase integral along the ray path.

This phenomenon of beam displacement is the basis for modified ray analysis considered in this paper. The development of the theory is given in Refs. 2 to 5 (the nomenclature is somewhat altered for convenience from the symbols used in these references).

The problem in obtaining quantitative results lies in the determination of the reflection coefficient

$$R(\theta_0) = |R|\, e^{i\varkappa(\theta_0)} \qquad \text{[Eq. 1]}$$

for realistic sound speed profiles. In particular, the modifications of ray analysis are especially significant for rays with turning-points, or vertices, near boundaries, or near a local maximum in sound speed.

To determine $R(\theta_0)$ we use an extension of the geometrical-optics approximation in a form that can be made valid throughout regions near a vertex on a ray. This procedure does leave some freedom in the choice of the sound speed profile $c(z)$ other than those for which exact solutions are known.

The method is such that the phase, $\arg R(\theta_0) = \varkappa(\theta_0)$, and the amplitude $|R(\theta_0)|$, and many features of the modified ray analysis can be described entirely in terms of a parameter $E(\theta_0)$ determined by the integral,

$$E(\theta_0) = i\frac{2}{\pi}k_0 \int_{z_1(\theta_0)}^{z_2(\theta_0)} \sqrt{n^2(z) - \sin^2\theta_0}\,dz \qquad \text{[Eq. 2]}$$

where $n(z) = c_0/c(z)$ is the index of refraction referred to a reference sound speed $c_0 = c(z_0)$. The angle θ_0 is measured with respect to the vertical, or z-direction, at a point on the ray at reference depth $z = z_0$. The symbol k_0 is a reference wave number, $k_0 = 2\pi f/c_0$, where f is the frequency. (We are considering a formalism for a harmonic source, but we can extend the application of the results to other time functions by the use of Fourier analysis). The limits of the integral are zeros, z_1 and z_2, of the integrand for a given value of θ_0. The reason for the appearance of two zeros, z_1 and z_2, when a ray in reality has only one real vertex, say at $z = z_1$, is indicated in the sketch in Fig. 1. It turns out that modified ray results are most significant when $c(z)$, or $n(z)$, is such that there are at least two zeroes of the integrand in Eq. 2. Therefore in this paper we consider profiles with a local maximum in sound speed, or where a boundary at $z = z_b$, as sketched above, gives rise to an "image" of the vertex at $z = z_1$. We confine our discussion to the three kinds of problems:

1) Zeros near a local maximum in sound speed (unbounded problem).

2) Zeros near a pressure release boundary.

3) Zeros near a rigid boundary.

The results for the phase $\varkappa(E)$ of the "plane wave" reflection coefficients for these three problems are as follows [see Ref. 4].

$$\varkappa(E) = -\chi(E) - \frac{\pi}{2} + \varepsilon \arctan e^{\frac{\pi E}{2}} \qquad \text{[Eq. 3]}$$

$\varepsilon = 0$ unbounded

$= -1$ pressure release boundary

$= +1$ rigid boundary

where the spatial function $\chi(E)$ is given by the expression

$$\chi(E) = \frac{E}{2} - \frac{E}{2} \ln \frac{|E|}{2} + \operatorname{Im} \ln \Gamma \left(\frac{1 + iE}{2}\right) . \qquad \text{[Eq. 4]}$$

The symbol $\ln|E|$ refers to the natural logarithm of the absolute value of E. The last term is the imaginary part of the logarithm of the gamma function [see Ref. 6, Ch. 6].

In Fig. 2, these phases are plotted as a function of the parameter E. Large positive values of this ray parameter E correspond, for the bounded problems, to rays that hit the surface, and the phases approach the limits we usually associate with such boundaries when the bounded medium is homogeneous ($-\pi$ for pressure release, 0 for rigid boundary). On the other hand, large negative E-values correspond to rays with vertices far from the boundaries, and all curves for phase approach the value $-\pi/2$, the phase change for plane waves reflected in a smooth monotonic profile.

The curves are continuous and not the simple "patched-up" step function that would result if we assume $\varkappa = -\pi$ for all rays hitting the surface, or $-\pi/2$ for all rays with vertices.

The amplitude of the reflection coefficient is, of course, unity for the bounded problems, while for the unbounded problem it has the following relatively simple form,

$$|R(E)| = \frac{1}{(1 + e^{\pi E})^{1/2}} \quad . \qquad \text{[Eq. 5]}$$

The formula for the displacement ΔR, of a ray that has transited a "two-turning-point" region is determined from the derivative of $\varkappa$ with respect to $\sin \theta_0$ [see Ref. 2],

$$\Delta R = -\frac{1}{k_0}\frac{d\varkappa}{d \sin \theta_0} = -\frac{1}{k_0}\frac{dE}{d \sin \theta_0}\frac{d\varkappa}{dE} \quad . \qquad \text{[Eq. 6]}$$

The derivative $d\varkappa/dE$ can be expressed as follows:

$$\frac{d\varkappa}{dE} = -\frac{d\chi}{dE} - \varepsilon \frac{\pi}{4} \operatorname{sech} \frac{\pi E}{4} \qquad \text{[Eq. 7]}$$

where $d\chi/dE$ can be written in the form

$$\frac{d\chi}{dE} = -\frac{1}{2} \ln \frac{|E|}{2} + \frac{1}{2} \operatorname{Re} \psi \left(\frac{1 + iE}{2}\right) \quad . \qquad \text{[Eq. 8]}$$

The last term involves the real part of the digamma function ψ [see Ref. 6, p. 258].

Since $\varkappa(E)$, $d\varkappa/dE$, and $|R(E)|$ are functions of E only, then the preparation of computer programs for the contribution of these functions to modified ray analysis is relatively simple. We only need to program their evaluation as functions of E. When the actual sound speed profile is selected then $E(\theta_0)$ and the derivative $dE/d(\sin \theta_0)$ needed for ΔR can be evaluated separately.

In our current programs, which have been written for the Woods Hole Oceanographic Institution's SIGMA 7 computer, and Hewlett-Packard computer, a composite linear/parabolic profile for $1/c^2$, or equivalently, for $n^2(z)$, is used. The function $n^2(z)$ and its first derivative are made continuous at some "matching" distance ρ, as sketched in Fig. 3.

In the sketch at the bottom of Fig. 3, for a more or less realistic profile, the regions in which "two-turning-point" phenomena may occur are shaded. Perhaps these programs may eventually be useful as simple subroutines added to already existing ray programs. When some "flag" tells a ray it has transited such a region, the subroutine supplies a displacement. The ordinary ray program in its usual way then continues the displaced ray until it again reaches a "diffraction" region. If the diffraction region is a region of local sound speed maximum the subroutine also supplies a change in amplitude due to leakage and splitting of rays.

The functions $\chi(E)$ and $d\chi/dE$ are the only terms in $\varkappa(E)$ and $d\varkappa/dE$ requiring special procedures for machine computation.

With series expansions for the gamma and diagamma functions [see Ref. 6, Ch. 6] inserted into Eqs. 4 and 8, $|\chi|$ and $d\chi/dE$ can be put into the form

$$|\chi| = \frac{1-\gamma}{2}|E| - 2|E|\ln 2|E| + \sum_{m=0}^{\infty}\left(\frac{|E|}{2m+1} - \arctan\frac{|E|}{2m+1}\right) \qquad \text{[Eq. 9]}$$

and

$$\frac{d\chi}{dE} = -\frac{1}{2}\ln 2|E| - \frac{\gamma}{2} + |E|^2 \sum_{m=1,3,5..}^{\infty} \frac{1}{m(m^2+|E|^2)} \qquad \text{[Eq. 10]}$$

where γ is Euler's constant,

$\gamma = 0.57721 \ldots\ldots$

The logarithmic terms in Eqs. 9 and 10 lead to computational difficulties since they become infinite for $E = 0$, that is for the grazing ray in the bounded case, or the split ray in the unbounded case. However, these infinities rather than being troublesome are one of the most physically significant contributions of the modified ray analysis. The following illustration shows why. Consider a purely parabolic profile for $n^2(z)$, for example,

$$n^2(z) = a + bz^2 \quad . \qquad \text{[Eq. 11]}$$

For a source and receiver at depths z_0 below the profile extremum, the ordinary ray theory result for the range R_{ray} of the ray emitted at angle θ_0 and connecting source and receiver, is given by the relation

$$R_{ray} = \frac{2 \sin \theta_0}{\sqrt{b}} \ell n \frac{[z_0^2 - z_1^2]^{1/2} + z_0}{z_1} \qquad \text{[Eq. 12]}$$

where z_1 is the depth of the ray vertex below the extremum of the n^2-profile. Note the logarithmic infinity for the grazing or split-ray $(z_1 = 0)$. However, for the profile of Eq. 11, the integral for E can be evaluated to give,

$$E = -k_0 \sqrt{b} \; z_1^2 \qquad \text{[Eq. 13]}$$

and, in turn,

$$-\frac{1}{k_0} \frac{dE}{d \sin \theta_0} = \frac{2 \sin \theta_0}{\sqrt{b}} \quad . \qquad \text{[Eq. 14]}$$

In consequence, it follows that the logarithmic term in the displacement is of the form

$$\Delta R = \frac{2 \sin \theta_0}{\sqrt{b}} \ell n (k_0^{1/2} \; b^{1/4} \; z_1) + \text{other terms} \quad . \qquad \text{[Eq. 15]}$$

Therefore, in the modified range R_{MOD} defined by the relation

$$R_{MOD} = R_{ray} + \Delta R \qquad \text{[Eq. 16]}$$

the logarithmic singularities cancel leaving a finite, wavelength-dependent range. For machine computation, therefore, we remove the logarithmic singularities from the displacement ΔR and R_{ray} before programming.

This feature of the modified ray theory that would arise for all smooth profiles (continuous derivatives in the region of the profile extremum), leads to the result that there is a split-beam shadow at finite range when modifications are included. Without the modifications every point is illuminated by a ray.

On the other hand, for a bilinear profile (discontinuous derivatives) ordinary ray theory does give a split-beam shadow at finite range, that is to say, there is no singularity in R_{ray}. For "non-smooth" profiles (discontinuous derivatives) we can also develop a modified ray analysis [see Ref. 4]. It turns out that in this case there is no singularity in the displacement ΔR. In fact, as we shall see, when the modified ray results are compared for a bilinear profile and a smooth profile where the smoothing compared with the bilinear case is only over a depth increment of a few wavelengths, the results are approximately the same. The modifications introduced by modified ray theory have removed some of the pathological sensitivity of ordinary ray analysis to small changes in the sound speed profile.

For the presentation of the results of the computer program, we currently use the format sketched in Fig. 4. The cycle range $R(\delta)$ for source and receiver at the same depth is plotted as a function of the grazing angle δ measured at the source point on a ray.

In Fig. 5 the solid curves represent examples for the three kinds of problems (1) unbounded, (2) pressure release boundary, and (3) rigid boundary. In the small figure in the upper left-hand corner, the shape of the R vs δ curve based on unmodified ray analysis is sketched; of course it is the same for all three problems, but different for the bilinear and composite profiles. Note the singularity in the ordinary ray theory results for a smooth profile. Note also that for the ordinary ray theory result for the bilinear profile the extremum in range would not be considered a caustic, in fact the derivatives of the R vs δ curve are discontinuous for this extremum (grazing ray).

The following conclusions can be drawn from the results presented in Fig. 5:

1) The modifications have removed the singularity in the R vs δ curves for smooth sound speed profiles. The resulting R_{MOD} vs δ curves are smooth so that the extremum does represent a caustic.

2) The modifications also lead to smooth R_{MOD} vs δ curves for the bilinear profile, so again the extremum actually is a caustic.

3) The derivatives of the R_{MOD} vs δ curves can be found and used as parameters in quantitative descriptions (Airy integrals) for the field in the neighbourhood of this caustic. Pedersen [Ref. 7] has shown how false caustics can arise in ordinary ray analysis for linear-segmented profiles. It is rather interesting to see that with modified ray analysis, the modifications not only may act to remove some of this pathology, but also may result in real caustics at places not recognized as caustics in the ordinary ray analysis for bilinear, or bounded linear $n^2(z)$ profiles.

4) The caustics are displaced in range compared with the location we might anticipate on the basis of ordinary ray analysis. For rays with vertices near a pressure release boundary, displacements are toward the source; for a rigid boundary, displacements are mostly away from the source.

5) The modifications have made the results different for all three kinds of problems, even for rays whose vertex would lie below the region where the models differ.

6) For the bilinear profile, and for a bilinear profile "smoothed" over a depth increment of "half-width" $\rho = 2$ wavelengths, the results are approximately the same.

In the lower part of Fig. 5, the values of the magnitude of the reflection coefficient for the unbounded problem are plotted to give quantitative results for the splitting of the rays into reflected and transmitted branches. The diffraction regions can be sources of "leakage" rays and diffraction reflected rays (from rays that according to ordinary ray theory would be totally transmitted).

In some of the examples we have analysed, it appears that for frequencies of the order of a kiloHertz the displacements can be hundreds of feet; while at lower frequencies, of 100 Hz or so, displacements amount to thousands of feet.

Finally, we would like to point out that whereas here we have considered a harmonic source represented as an integral over ray parameter θ_0, it is also possible to consider signals of different time behaviour, Fourier-analysed so that the integral is over the variable ω. In that case, we can also show that there are time delays or advances Δt, in arrival times, that are related to derivatives of the phase with respect to ω. It can be shown that these time displacements can also be written in terms of the derivative $d\varkappa/dE$, as follows,

$$\begin{aligned}\Delta t &= \frac{d\varkappa}{d\omega} = \frac{d\varkappa}{dE}\frac{dE}{d\omega} \\ &= \frac{E}{\omega}\frac{d\varkappa}{dE} .\end{aligned} \qquad [Eq.\ 17]$$

Recently, a number of laboratories have been using machine programs for modal analysis where a sufficient number of modes (perhaps a thousand, if necessary) are summed to describe the sound field for comparison with ray computations. It should be borne in mind, when making such comparisons, that the modal analysis is a more complete and exact analysis for the field, and if there is any validity for the displacements or modifications we have described in this paper, the modal results should already include such effects.

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DISCUSSION

In answer to questions the first author said that, broadly speaking, modifications to ray analysis were needed if the ray has a vertex within 4 or 5 acoustic wavelengths of a surface; and that the linear/parabolic model with "half-width" ρ was chosen for purposes of illustration.

The first author affirmed that amplitudes could be obtained straight-forwardly from modified ray analysis. In reply to a suggestion that this was equivalent to including higher order terms in an inverse-wavenumber expansion, the first author said that this may be so, but that the individual terms were not the same as in the traditional expansion.

Some discussion at this session, and in later private sessions, revealed that the techniques of modified ray analysis might be usefully applied to remove some of the pathological aspects of discontinuities when velocity profiles are approximated by constant-gradient segments.

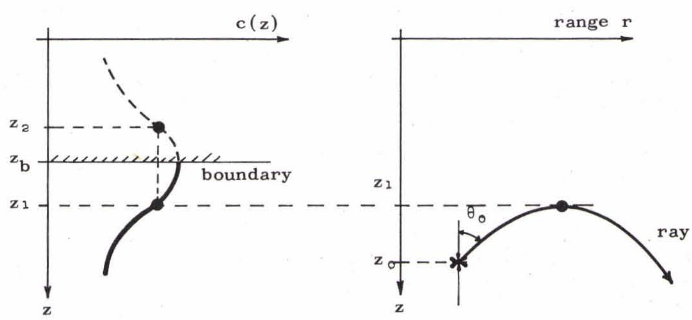
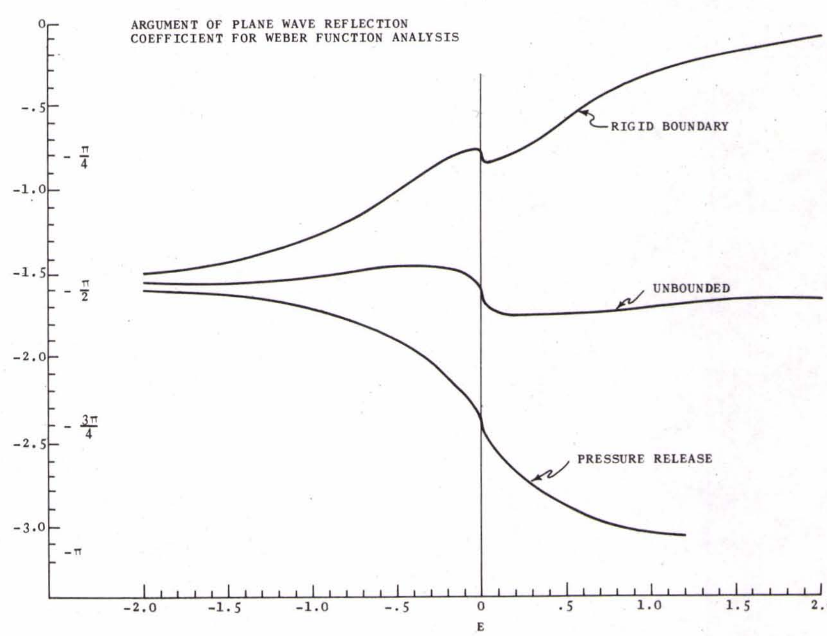


FIG. 1

FIG. 2



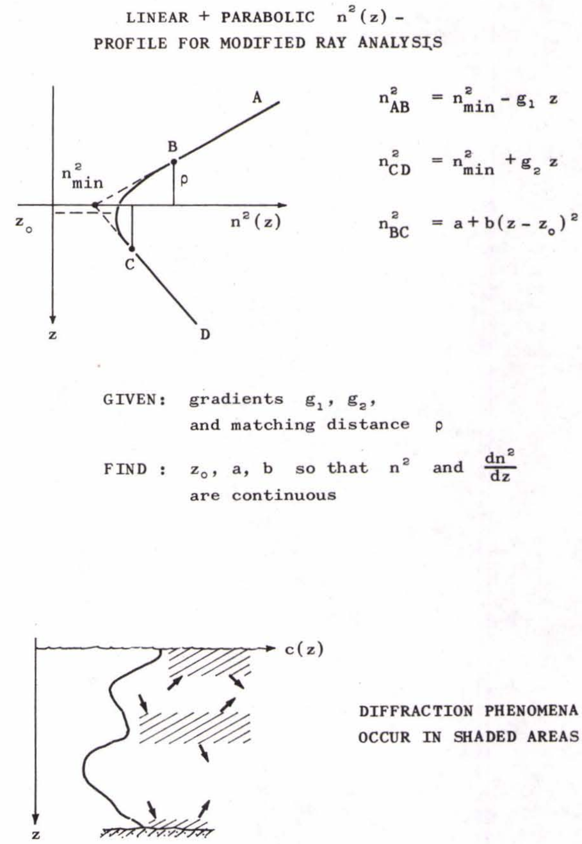


FIG. 3

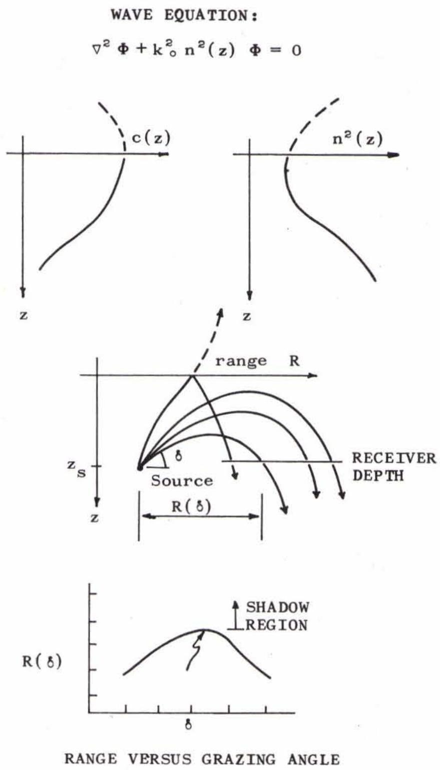


FIG. 4

FIG. 5